Influence Diagrams: A New Approach to Modelling Games

by

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Contents

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Declaration

I hereby declare that this thesis is entirely my own work, with the exception of sections 6.3.3 and 7.2.2, which result from joint work with my supervisor, Jim Smith.

Summary

Game theory seeks to describe the interaction of two or more actors with distinct objectives. This is achieved using a mathematical model known as a game. Virtually all game theory relies on either the extensive form or the normal form to represent the games being studied.

By drawing on the previously unrelated fields of game theory and graphical modelling, and by taking a new approach to the way in which a game is modelled, an alternative to the extensive and normal forms is developed: the belief influence diagram (BID). Starting from the basic definition of a game and using a new form of conditional belief called a prospective function, it is shown how the decision influence diagram can be adapted to model games.

The advantages of the BID over the extensive and normal forms are explored, particularly its ability to model some of the qualitative aspects of games and to model games of greater complexity. By using BIDs in the modelling of games, fresh insight can be gained into certain features of the game, such as what sources of information an actor in the game should take account of.

New concepts of sufficiency and parsimony are defined which relate to the BID. It is shown how these concepts, when combined with different forms of rationality, can lead to a variety of methods for simplifying a BID, and hence simplifying the game which it represents. It is shown that such simplifications are invariant with respect to the order in which the simplifying steps are carried out.

A schematic version of the BID is used to model finite repeated games and to develop concepts of learning and local sufficiency. It is shown how BIDs can be used to facilitate an induction proof in a finite repeated game and to model a highly complex competitive market. This last example is used to illustrate how BIDs can be helpful in evaluating some qualitative aspects of a model.

1 Introduction

Game theory seeks to describe the interaction of two or more actors with distinct objectives. This is achieved using a mathematical model known as a game. Since the theory of games depends to a great extent on the form taken by those models, a new approach to modelling implies a new approach to game theory itself.

Game theory is dominated by the use of two types of model: the game tree, or extensive form, and the payoff matrix, or normal form. Virtually all game theory relies on these forms, or variants thereof, to represent the games being studied.

By drawing on the previously unrelated fields of game theory and graphical modelling, and by taking a new approach to the way in which a game is modelled, we develop an alternative to the extensive and normal forms: the belief influence diagram (BID). We demonstrate some of the advantages of the BID over the extensive and normal forms, particularly its ability to model some of the qualitative aspects of games and to model games of greater complexity. We show how various assumptions about the rationality of actors can allow us to simplify the BID.

We take the view that a model of a game must capture the perceptions of the actors playing that game. Therefore our theory is based on the personal beliefs of those actors, and thus is subjective in nature. However, the theory is not 'Bayesian' as such, since another intention is to make the theory as general as possible. In fact, the model which we derive can be formulated to fit any subjective framework, including the 'Bayesian', as a special case.

Chapter 2 is a general survey of the game theory literature, and maps out the development of some of the more important ideas, as well as some of the paradoxes and problems with the theory. Chapter 3 looks at the literature on graphical modelling, giving an overview of the various types of graphical model, and concentrating on the use of the influence diagram in decision theory.

In chapter 4, we start from the foundations of game theory and devise step by step a new modelling framework for games. We describe a form of conditional belief known as a prospective function, from which arises a new version of conditional independence, called belief conditional independence. The belief influence diagram is defined according to the properties of belief conditional independence.

In chapter 5, we consider in detail the forms of rationality which an actor may adopt. We derive a theory of sufficiency as it applies to the BID. This is linked to the rationality of actors through a principle of parsimony. In chapter 6, we show how various forms of rationality may allow us to simplify the BID (and hence the game itself) by deleting certain nodes and arcs. We prove that the simplification of a BID is invariant with respect to the order in which these deletions take place. We also consider the implications which rationality has for the principles of modelling a game.

In chapter 7, we take a look at finite repeated games, and use their special structure to generate the ideas of local sufficiency and local optimality. We consider a definition of learning in repeated games, and demonstrate how the ideas of the previous chapters can be applied. We show how BIDs can assist in proving a general theorem about finite repeated games using induction. As an example, we demonstrate how a highly complex competitive market may be modelled using BIDs, and consider briefly how BIDs can be helpful in evaluating some qualitative aspects of a model.

In chapter 8, we present our conclusions, and some suggestions for further research on the theory of games developed in this thesis.

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In chapter 8, we present our conclusions, and some suggestions for further research on the theory of games developed in this thesis.

2 A Brief History of Game Theory

2.1 Foundations

Game theory seeks to describe the interaction of two or more actors with distinct objectives. This is achieved using a mathematical model known as a game.

2.1.1 The origins of game theory

The earliest examples of such mathematical models occurred in economics, most notably in the theory of oligopoly, as developed by Cournot in the 19th century.

The first to regard games as models to be studied in their own right was Borel (1921). He considered the concepts of pure and mixed strategies as they apply to what are now known as normal form games. However similar ideas were independently pursued by von Neumann in the 20's and 30's, and it was the publication in 1944 of von Neumann and Morgenstern's "Theory of Games and Economic Behaviour" which established game theory as a recognised field of academic study.

This remarkable work contained many of the ideas which would be employed by game theorists in future years, including an axiomatic definition of a game, derivation of extensive and normal forms, pure and mixed strategies and the minimax theorem. It also predicted many of the areas where advances would be made. Indeed all subsequent work on game theory can be traced back to it.

2.1.2 The extensive form

The version of the extensive form which is used today is due to Kuhn (1953). The *extensive form* of a game consists of a tree wherein each non-terminal node represents an action to be taken by one of the actors, and each arc out of a node represents one of the possible acts available to that actor. Chance actions can be represented by designating one of the actors as 'nature' and including her actions in the same way. Each terminal node represents the outcome resulting from the sequence of acts represented by the path from the origin.

Associated with each outcome is a utility vector or *value*. Von Neumann and Morgenstern (1944) defined an actor's utility according to axioms of expected utility using frequentist probabilities. This was generalised to include subjective probabilities by Savage (1954) for decision theory and by Luce and Raiffa (1957) in the case of game theory.

In a game of *perfect information* every actor knows the complete history of the game whenever he takes an action. In the case of *imperfect information* an actor may be uncertain as to which of two or more nodes represents the action he is about to take. A set of nodes between which an actor is uncertain as to which represents his action is said to form an *information set*. If an information set contains two or more nodes, they are connected by a dotted line in the extensive form.

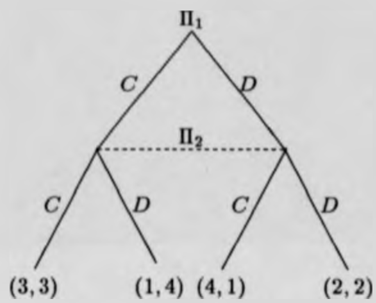


Figure 2.1. The Extensive Form of a Prisoner's Dilemma Game

Thus in the game depicted in Figure 2.1, actor Π_1 chooses whether to co-operate (C) or defect (D). In ignorance of Π_1's act, Π_2 faces the same choice. The game shown here is known as a Prisoner's Dilemma Game (PDG), as first described by A.W. Tucker.

2.1.3 Strategies and the normal form

Clearly as the number of moves by each actor increases, the extensive form will become unmanageably large. An alternative method of describing a game is to consider the strategies of each actor. A *(pure) strategy* for an actor consists of a sequence of acts, one for each information set at which he may have to take an action. Thus it represents a complete specification of how he will play a game whatever is done by the other player(s).

So instead of an actor choosing an act whenever one of his information sets is reached, he could simply choose one strategy. If every actor does this, it determines the outcome and value of the game. The *normal form* of a game with n actors $\Pi_1, \ldots, \Pi_n$ is an n-dimensional matrix in which a plane perpendicular to the i-th dimension corresponds to a strategy of Π_i and each element is the value associated with the relevant outcome. It is usual to marginalise out any chance actions by taking expectations of the value over the appropriate probability distributions.

Thus the normal form for the game shown in Figure 2.1 is:

		Π_2	
		C	D
Π_1	C	3, 3	1, 4
	D	4, 1	2, 2

Figure 2.2. The Normal Form of a PDG

Compared with the extensive form, the normal form is particularly convenient for games with two actors and imperfect information, such as PDGs.

2.2 Equilibrium Theory

Much of game theory over the years has been concerned with advising actors on how they ought to play certain games. And right from the start, most authors have focussed on equilibrium as a key concept. An *equilibrium* in a game is a set of strategies, one for each actor, such that each strategy is an optimal response to the set of strategies played by the other actors.

Let us consider some of the reasons why equilibrium has assumed such an important role in game theory. The first is that the primary motivation for game theory arose initially out of some problems in economics, in which equilibrium was already a familiar idea. Also the earliest games to be studied in detail were predominantly two-actor zero-sum games, for which equilibrium is the most natural and justifiable outcome. Then there is the widely held belief that game theory should inform not just one actor in a game, but all actors, simultaneously. In other words, all actors should be told how to play the game optimally. It is clear that in order for such a theory to be self-consistent, the set of proposed optimal strategies must form an equilibrium.

2.2.1 Nash equilibrium

Von Neumann and Morgenstern (1947) focussed on zero-sum two person games. In the case of perfect information, Zermelo's Theorem (Zermelo 1912) shows that an equilibrium exists in pure strategies, and can be constructed from the extensive form via backwards induction. In the case of imperfect information, the existence of an equilibrium in pure strategies is not guaranteed, so mixed strategies are required.

A *mixed strategy* for an actor is a probability distribution over his set of pure strategies. Von Neumann's Minimax Theorem showed that all zero-sum 2-person games (of complete information) have an equilibrium in mixed strategies. This result was extended by Nash (1951) who proved that all finite n-person games of complete information contained at least one equilibrium, or *Nash Equilibrium*, as it would from then on be known. In fact, as is shown by Aumann and Brandenburger (1991), weaker conditions than complete information will suffice to guarantee a Nash Equilibrium.

2.2.2 Problems with mixed strategies

The use of mixed strategies is by no means uncontroversial. For a start, suppose the expected utility to an actor resulting from a unique equilibrium (in mixed strategies) is v. There are some instances where v can be guaranteed as a minimum by adopting one pure strategy but not another, even when the second is played with positive probability according to the mixed strategy. This was noted, among others, by Harsanyi (1964) who suggested that an actor faced with such a situation should not play such a mixed strategy. This result was extended by Aumann and Maschler (1972) who cast doubt on the validity of Minimax even in certain zero-sum games.

However there is a more fundamental objection to the use of mixed strategies, voiced by many over the years, namely whether it makes sense to play them at all. According to early (frequentist) definitions, a mixed strategy would result from the actor concerned carrying out in private some experiment for which the outcomes had known probabilities corresponding to those specified by the strategy, and then playing the pure strategy determined by the outcome of the experiment. Such a mechanism was seen to be inherently unsatisfactory by many, including Howard (1971) who questioned whether a 'rational' person would ever base an important decision on, for example, the toss of a coin.

To justify the concept of mixed strategies, it is necessary to consider the original motivation as advanced by Borel (1921) that an actor's strategy, if it became known to his opponent, would allow her in certain situations to take advantage of this knowledge, whereas a mixed strategy would keep her guessing. Thus it can be seen that the essence of a mixed strategy lies not in the explicit randomisation of play, but in the (subjective) uncertainty of an actor's opponents as to which *pure* strategy he is playing, as pointed out by (for example) Aumann (1987) and Binmore (1988).

The question then arises as to what happens in games with three or more actors, in which more than one actor has some uncertainty. The usual way of handling this is to impose the condition that the uncertainty is expressed in terms of a commonly held belief on the part of an actor's opponents. Such an interpretation is advanced by Rubinstein (1991) and Brandenburger (1992). Some of the consequences of relaxing this condition are explored by Aumann (1974).

2.2.3 Refinements of Nash equilibrium

For the last forty years Nash Equilibrium (N.E.) has been the dominant idea in game theory. One measure of it's success is the sheer volume of suggested alterations, by the tightening or relaxing of conditions, or the imposition of new conditions. To many, the principal difficulty with N.E. is that in general it is not unique. Thus it does not always provide a definitive answer to the question posed by a person playing the game: "How should I play?". Hence much effort has been expended in weeding out certain types of N.E. as being unsatisfactory, for one reason or another, in an attempt to see if a 'best' equilibrium or set of equilibria can be found.

Much of the disagreement on which criteria should be used to discriminate between alternative equilibria has centered on the argument over which of the normal form and extensive form is the more fundamental. Selten (1975) employs the extensive form to motivate his ***subgame perfect equilibrium*** concept. Using the idea of a "trembling hand" which induces small mistakes (ie. deviations from equilibrium strategy), he considers which N.E.s are self enforcing in the limit as the probability of mistakes tends to zero.

Kreps and Wilson (1982) define the almost identical concept of ***sequential equilibrium*** by considering explicitly the beliefs of actors who find themselves at a point in the extensive form which is off the equilibrium path. They argue that every act taken must be part of an optimal strategy for the remainder of the game. Myerson's (1978) *proper equilibrium* is related to perfect equilibrium, but demands in addition that the probability of a mistake be monotone decreasing in relation to the cost of the mistake. One problem with this is that the axiom of independence of irrelevant alternatives may be violated, as was noted by Jurada and Sanchez (1990).

On the other side, Kohlberg and Mertens (1986) argue that equilibria based on the extensive form may not be invariant under certain innessential transformations (for example the addition of a trivial action). They claim that the set of self-enforcing, or *strategically stable equilibria* should depend only on a 'reduced normal form' (derived by considering only equivalence classes of mixed strategies). A related concept is *persistent equilibrium*, as described by Kalai and Samet (1984).

Perhaps the most extreme example of equilibrium refinement is the tracing procedure of Harsanyi and Selten (1988). They argue that the use by an actor of a N.E. strategy cannot be justified unless it is unique or he knows all the other actors will choose strategies corresponding to the same equilibrium. They therefore devise a method for recommending a single equilibrium for any game, based on perfectness, payoff-dominance, risk-dominance and symmetry. They base their analysis on what they call the "standard form" of a game, which combines elements of both the normal and extensive forms.

2.2.4 Alternatives to Nash equilibrium

While attempts to reduce the number of equilibria have predominated, some authors have suggested that Nash Equilibrium is too narrow as a solution concept. For the outcome of a game to form a N.E. relies implicitly in many cases on all the actors believing that it will occur, prior to taking their actions (or choosing their strategies). Yet it can be argued that this assumption is under many circumstances unreasonable. Indeed, in a game where there is more than one N.E., the outcome may not be a N.E. despite all actors choosing a N.E. strategy.

Consider the normal form game shown in Figure 2.3, commonly called the "Battle of the Sexes".

In this game there are two pure strategy equilibria $(A;a)$ and $(B;b)$ and a mixed strategy

Π_2

Π_1	2, 1	0, 0
	0, 0	1, 2

Figure 2.3. The Normal Form for "Battle of the Sexes"

equilibrium $(\frac{2}{3}A, \frac{1}{3}B; \frac{1}{3}a, \frac{2}{3}b)$, with respective values $(2,1)$, $(1,2)$ and $(\frac{2}{3}, \frac{2}{3})$. Now suppose we alter the game so that both actors observe the outcome of some random device, say tossing a fair coin. Then the strategies $(A;a)$ if heads, $(B;b)$ if tails form a new equilibrium with value $(\frac{3}{2}, \frac{3}{2})$. Note that this equilibrium ***pareto-dominates*** the old mixed strategy equilibrium, that is it results in a higher utility for both actors.

This is an example of a ***correlated equilibrium***, as defined by Aumann (1974). In Aumann (1987), he shows that under certain assumptions, a normal form game played by "rational Bayesians" will always result in such an equilibrium. Note that the set of correlated equilibria includes as special cases all Nash equilibria.

A quite different approach is taken by Bernheim (1984) and Pearce (1984), who contend that it is not just Nash Equilibrium but the concept of equilibrium itself which is too restrictive. They concentrate instead on the types of strategies which "rational" actors might reasonably play, using the idea of iterated dominance.

A strategy is said to ***dominate*** another if it results in a higher utility for the actor concerned whatever the other actors do. This is equivalent to the "Sure-thing" principle of Savage (1954). Playing a dominated strategy is ruled out under all the equilibrium concepts discussed so far.

Assuming that no actor will ever play a dominated strategy, the corresponding rows or columns in the normal form can be ignored. This may lead to previously undominated strategies available to other actors becoming dominated. Thus it is reasonable to assume those will never be played. This process, known as ***iterated dominance***, continues until only undominated strategies remain. These strategies are called ***rationalizable*** by Bernheim and Pearce. Playing of rationalizable strategies does not in general result in an equilibrium, although Brandenburger and Dekel (1987) give an interpretation of rationalizability which shows the connections with equilibrium and correlation.

2.3 The Role of Information

The information which an actor has when playing a game and the effects of that information have received increasing attention over the years. Among the most important features to be studied are common knowledge, learning in repeated games and the case of incomplete information.

2.3.1 Common knowledge

When playing a game of any complexity, an intelligent actor will usually ask himself the question: "What do I know about my opponent(s)?" Of particular interest is the state of mind of an opponent, including the question: "What do my opponents know about me?" Such considerations lead to statements of the kind: "I know that you know that he knows that she knows that ... X", and so on ad infinitum. Roughly speaking, if all statements taking this form are true (for any finite sequence of actors with no two adjacent ones the same), then X is said to be common knowledge.

Common knowledge is one of the main foundational concepts which underlies game theory. Indeed, before any analysis of a game can take place, the common knowledge base must be established. This will always include some definition of the 'rules of the game', even if this is simply the order in which actions are taken.

For example, all of the previous discussion implicitly uses common knowledge, such as common knowledge of the normal or extensive forms or of an actor being "rational". Aumann (1976) was the first to consider common knowledge in detail and give a formal definition, to which the following is equivalent.

Definition 2.1 *Let $\Pi_K = \{\Pi_{(1)}, \ldots, \Pi_{(k)}\} \subseteq \Pi$ be a set of actors. Define $K(\Pi_K)$ to be the common knowledge set for Π_K. Then for any statement 'X',*

$$\text{'}X\text{'} \in K(\Pi_K) \iff \Pi_{(i)} \text{ knows } X\text{'} \in K(\Pi_K) \text{ for } i = 1, \ldots, k.$$

We say 'X' is common knowledge *among the actors in Π_K.*

Milgrom (1981) provides an axiomatic characterisation of common knowledge. Brandenburger (1992) analyses the connections between common knowledge and various solution concepts, Geanakoplos (1992) discusses the implications for certain problems outside game theory and Reny (1992) considers what happens when common knowledge breaks down. A good review can be found in Binmore and Brandenburger (1990).

2.3.2 Paradoxes of rationality

What does it mean for an actor to be "rational"? I suppose just about everyone would agree that it has something to do with utility maximisation. But beyond that, consensus breaks down. Consider the various solution concepts outlined above. Each, either explicitly or implicitly, relies for justification on some definition of rationality. Indeed, each must rely on a different definition, for how else could differing outcomes obtain?

A cynical interpretation might be that rationality is in some cases defined in terms of the solution concept, with Bernheim (1984) and Pearce (1984) coming closest to admitting this by the use of the term "rationalizable" to describe their concept. Now if anyone is justified in laying claim to a general condition of rationality then they are, since they rely solely on iterated domination. Yet even domination turns out not to be such a "sure thing" as it appears.

Consider the PDG shown in Figures 2 and 3. The only equilibrium is (D, d), moreover D dominates C and d dominates c. However, (C, c) pareto-dominates (D, d), a phenomenon which has caused more to be written about PDGs than most other games put together. This phenomenon becomes more interesting when two actors play a number of PDGs consecutively, known as a repeated game.

In both the one-off PDG and the finitely repeated PDG all the solution concepts above recommend the playing of (D, d) at every stage. While this seems reasonable for a single PDG, many disagree when it comes to the repeated version. Howard (1971) uses the repeated PDG to illustrate his assertion that in general it is impossible for all actors to be what he calls "objectively rational". He suggested that a person playing such a game should cooperate in order to induce his opponent to do likewise. Such an approach is in fact well-founded, considering that in experimental games people often do better by adopting such strategies than by continually defecting. For a comprehensive review of the experimental literature in game theory, see Colman (1982).

A similar phenomenon can occur even in games of perfect information. Examples include Selten's (1978) "Chain-store Paradox", Rosenthal's (1981) "centipede", and the "Take it or leave it" game of Reny (1992). In each case, an actor can in some circumstances do better by rejecting the backward's induction analysis implied by Zermelo's theorem and adopting instead a form of "forward induction", involving inducement or (in the chain-store paradox) deterrence.

Such arguments tend to suggest that, while it is reasonable to suppose that rationality involves utility maximisation, there is in general no method of achieving this which can incontrevertibly be called "rational". Basu (1990) uses Rosenthal's centipede to show that under certain reasonable axioms, any definition of rationality for extensive games leaves open the possibility of a contradiction: that an "irrational" move leads to a higher utility than a "rational" one.

Binmore (1987) uses such arguments to conclude that all rationality must be imperfect, or *bounded*. Bounded rationality involves an actor taking into account the cost of thinking about his choice of act, and the cost of thinking about the cost of thinking about his choice, and so on. Simon (1955) was the first to introduce the concept of bounded rationality. Alternative approaches are taken by Binmore (1988) and Lipman (1991).

2.3.3 Infinitely repeated games

As we have already seen, playing a game a number of times in succession can lead to outcomes quite different from those which occur in a one-shot game. Formally, a *repeated game* comprises a number of consecutive plays of a game, known as the *stage game* or *generating game*, by the same actors. The utility to an actor of the repeated game is a monotone non-decreasing function of the utility from each stage game. It is usually assumed that this takes the form of the sum of utilities for finitely repeated games, and either the sum with discounting or the limiting mean in the infinite case, although this assumption is questionable.

As with one-shot and finitely repeated games, the study of infinitely repeated games has resulted in a number of different approaches. Perhaps the best known is the Folk theorem, so called because it arose from work by several authors. Shubik (1970) illustrated the result for the special case of the PDG, which he attributed to R.J. Aumann, the general case being developed by Rubinstein (1979) and Fudenberg and Maskin (1986).

Informally, the Folk Theorem states that under general circumstances, every feasible outcome which does not reward any actor less than his minimax utility, can arise as a Nash equilibrium. The set of equilibria satisfying these conditions is in general uncountable — quite a contrast from the single equilibrium result of Harsanyi and Selten (1988). Figure 2.4 illustrates the set of equilibria for the infinitely repeated PDG.

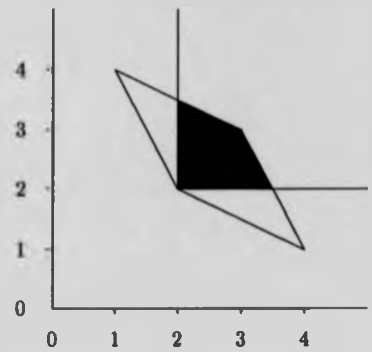


Figure 2.4. The Core of the Infinitely Repeated PDG

Other approaches to infinitely repeated games are taken by Howard (1971) and Grofman and Pool (1975). Fudenberg and Levine (1990) examine the case where an actor plays a stage game against a sequence of different opponents. Hart (1985) characterises the equilibria of nonzero sum two person repeated games, under conditions of incomplete information, to which topic we now turn.

2.3.4 Incomplete information

So far we have taken for granted that the basic structure of the game, including the acts or strategies available to any actor and the value resulting from every outcome, are known to all actors, indeed are common knowledge. This condition characterises games with *complete information*; a game which does not satisfy this condition is said to have *incomplete information*. For the most part such games were left alone by the early classical game theorists, yet they represent an important class of games, in so far as a game intended to model accurately a real life situation is likely to include elements of incomplete information.

The lack of common knowledge of certain aspects of a game leads to an infinite regress; each actor must explicitly consider his beliefs about his opponent, his beliefs about his opponent's beliefs about him, and so on, in an infinite hierarchy of beliefs. The first to tackle this problem in a systematic way was Harsanyi (1967). He modelled a game of incomplete information as having resulted from a prior distribution which was common knowledge. Each actor's utility and information is considered to be the result of a lottery distributed according to this common prior distribution. Thus a game with incomplete information can be converted to a "Bayesian game" with complete (but imperfect) information which is tractable. Mertens and Zamir (1985) give a mathematical formulation of this.

Harsanyi (1968a) justifies the common prior assumption (now know as the "Harsanyi doctrine") by claiming that all differences between actors can at some level be explained entirely by differences in experience or information. His analysis has been utilised by many, including Aumann (1987). However it has many critics (see for example Binmore (1991) and Gul (1991b)).

An alternative approach to a game of incomplete information (and in the case of some authors, to a game of complete information) is to represent it as a decision problem from the point of view of each individual actor. The analysis then follows along the lines proposed by Savage (1954). The actor evaluates his prior distribution for every unknown element of the game, including the nature of his opponents. He then acts so as to maximise his utility subject to these beliefs, updating them as the game progresses according to the standard Bayesian analysis. Advocates of this subjective Bayesian approach include Kadane and Larkey (1982, 1983), Tan and Werlang (1988) and Rosenthal (1981).

The approach based on solution concepts is criticised (by Kadane and Larkey (1982) and Binmore (1987) among others) for assuming too much: that it is common knowledge that all actors are rational. The subjectivist approach is criticised (by Harsanyi (1982) and Binmore (1991)) for assuming too little: that there are no restrictions on what beliefs an actor may adopt. The consequences of some possible exogenous restrictions on beliefs are discussed in Gul (1991a) and Brandenburger (1992).

This represents one of the most fundamental disagreements in game theory, although there have been attempts (see for example Nau 1992) to reconcile the two approaches. For the record, I would recommend a pragmatic approach. When playing a game one has at some point to make a decision, and in that sense an actor should base any decision on his (subjective) evaluation of the situation. Such an evaluation should however be informed by theory relevant to the behaviour of actors in a game, which may well include the implications of solution concepts.

2.4 Back to Basics

Underlying this disagreement is the distinction between *normative* or *prescriptive* theory which sets out what "rational" actors should do, and *positive (descriptive, predictive)* theory which attempts to describe what actually happens. Most of the non-experimental game theory literature, such as is discussed above, concerns the former, Howard (1971) being one exception.

It is claimed by Kadane and Larkey (1983) that one of the major problems with game theory is that the two types of theory are so often confused. One of the consequences is that a normative theory which attempts to advise all actors in a game (in other words a solution concept) bases its advice to each actor on the assumption that every other actor plays the game in a way which is recommended by the theory. Hence a normative theory depends on the positive theory that there is no difference between the normative and the positive.

This might not be such a terrible problem were it not for the diversity of normative theories, none of which has been able consistently to predict how people will play a game. It seems the only way in which this might occur is the adoption of a certain way of playing by substantial numbers of people as the result of the publication of a theory recommending it, and there is no real evidence that this has ever happened.

Yet there is an even more fundamental objection to the solution concept approach. This relates to the way in which games are modelled. Now there are two possible ways in which a situation of conflict can be analysed using a mathematical model. The first is to include every essential aspect of the situation in the model. Then the model can be analysed as if it were the real-life situation, and provided the model is reasonably accurate any result obtained should be directly relevant to the situation.

The second is to summarise the situation by incorporating into the model a relatively small amount of information. The analysis of such a mathematical model will not in itself suggest a course of action, but may be used constructively, by considering the results in the context of the real life situation. Now the first approach is difficult, since most models would quickly become far too complicated to be analysed. The second is in some sense unsatisfactory because it does not provide any definite answer.

So solution concepts cleverly sidestep this problem by using only a simple game model, and then analysing it as if it were in fact the real life situation. Binmore (1987, 1988, 1991) makes much the same criticism. He suggests as an alternative the use of an imperfectly rational decision criterion, which recommends a "best guess" if an optimal solution has not been within a fixed period of time.

Given these fundamental problems with the modelling of games, and given that the relative primacy of the normal and extensive forms is at best undecided, we must surely conclude that there is room at least to consider alternative models. Any rival to these long-established models must demonstrate an advantage in some of those areas in which they are deficient. One area of deficiency common to both is the problem of modelling anything other than the simplest of situations. One possible way of overcoming this deficiency is by using an influence diagram instead of a tree as the basis of our model. This is the approach I propose to take.

3 An Introduction to Graphical Modelling

3.1 The Elements

The modern theory of graphical models was formed by the fusion of two ideas: descriptive models based on the concept of causality and the statistical theory of conditional independence.

3.1.1 Early causal models

The year 1921, when Borel published his ideas on game theory, also saw the first developments in the theory of graphical modelling. The concept of path analysis was introduced to statistics by the geneticist Wright (1921). It was based on the construction of a qualitative diagram, called a *path diagram* in which the variables were linked by directed arcs indicating causality and correlation.

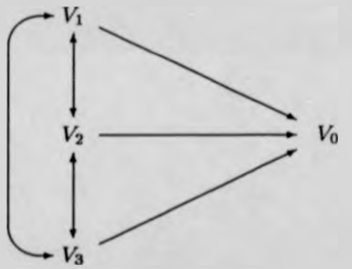


Figure 3.1. An Early Path Diagram

In figure 3.1, the random variable V_0 has directed edges from the causal variables V_1, V_2 and V_3. They, in turn, are linked to each other by double-headed arrows to signify unknown degrees of correlation between them.

The path analysis rules of Wright (1921, 1934) provided a method of measuring influence along each path in the diagram, and of finding the degree to which an effect was determined by a particular cause. This method was limited to the analysis of linear relations between continuous random variables.

Path diagrams and other related models have found many uses. Apart from work on measuring correlation and causality (see Kiiveri et al, 1984, for a review of the early work in this area), a variety of causal models have been employed in economics and the social sciences for both quantitative and qualitative analysis; see, for example, Blalock (1971).

However, for many years a general theory combining path analysis with probability calculus proved elusive, due it seems to the complicated nature of the possible relationships which could obtain between several variables. Goodman (1973), who analysed similar graphs involving discrete variables, was one of the first to notice a link between the structure of a path diagram and the presence of conditional independence constraints on the distribution of the random variables. This, when combined with Dawid's (1979) axiomatisation of conditional independence, provided the necessary analytic tool to put graphical models on to a firm mathematical footing.

3.1.2 Conditional independence

Probabilistic conditional independence, or conditional independence (c.i.), is a relationship $\cdot \perp\!\!\!\perp_P \cdot | \cdot$ defined over three random vectors, or sets of random vectors.

Definition 3.1 *Let Q be a set of random vectors, and let $X, Y, Z \subseteq Q$. Then we interpret the relation $X \perp\!\!\!\perp_P Y | Z$ to mean: "X is (probabilistic)* conditionally independent *of Y, given Z." or "Given Z, Y is uninformative about X." In the case that X is (unconditionally) independent of Y we write: $X \perp\!\!\!\perp_P Y | \emptyset$, or just $X \perp\!\!\!\perp_P Y$*

Dawid (1979) demonstrated the importance of conditional independence in statistics, and derived several properties of probabilistic c.i. The most important of these are summarised (using a general conditional independence notation) by the properties C1–C3 below. These have since been adopted as the standard c.i. axioms by most researchers in the field, and are sometimes called the *semi-graphoid* axioms (Pearl and Verma 1987) or *graphoid* axioms (Geiger et al 1990).

For $W, X, Y, Z \subseteq Q$,

C1 $X \perp\!\!\!\perp Y | Y \cup Z$, .

C2 $X \perp\!\!\!\perp Y | Z \iff Y \perp\!\!\!\perp X | Z$.

C3 $X \perp\!\!\!\perp Y \cup Z | W \iff X \perp\!\!\!\perp Z | W$ and $X \perp\!\!\!\perp Y | Z \cup W$.

Other axioms proposed include the 'intersection' axiom,

C4 $X \perp\!\!\!\perp Y|Z$ and $X \perp\!\!\!\perp Z|Y \implies X \perp\!\!\!\perp Y \cup Z|Y \cap Z$.

However this does not hold in general for probabilistic c.i., except under positivity or similar constraints. It is also the case, according to Dawid (1993), that probabilistic c.i. cannot be characterised by a finite set of axioms (although C1–C3 appear to be adequate for most purposes).

Smith (1988) argued that C1–C3 ought to apply to systems which incorporate objects other than random vectors. He postulated that any sensible concept of relationship between variables should exhibit such a c.i. structure. He defines (Smith 1989b) *generalised conditional independence* to be any relation of the form $\cdot \perp\!\!\!\perp \cdot | \cdot$ which satisfies axioms C1–C3.

A number of versions of conditional independence as presented by Dawid (1993) are described briefly below:

Linear C.I. (Smith 1990): Let Ω be a linear space of variables, with covariance Σ. Then $X \perp\!\!\!\perp_L Y|Z(\Sigma)$ if the coefficient of Y in linear regression of X on (Y, Z) can be taken as 0 (or equivalently, X and Y have zero partial correlation with respect to Z).

Conditional Variance Independence: Let S be any subset of the sample space Ω. Then $X \perp\!\!\!\perp_V Y|Z[S]$ if $\{X(\omega) : \omega \in S, Y(\omega) = y, Z(\omega) = z\}$ depends only on z.

Meta-C.I.: Let $\mathcal{P}$ be a parametrised family of distributions. Then $X \perp\!\!\!\perp_M Y|Z[\mathcal{P}]$ if $X \perp\!\!\!\perp_P Y|Z[\theta]$ for all $\theta \in \mathcal{P}$ and $\theta_{(X,Z)} \perp\!\!\!\perp_V \theta_{(Y,Z)}|\theta_Z[\mathcal{P}]$, where $\theta_{(\cdot)}$ is the value of θ which parametrises the (joint) marginal distribution over $(\cdot)$. Smith (1990) described a special case of meta-c.i., known as *parametrised family* c.i.

Hyper-C.I.: Let Π be a prior distribution for θ. Then $X \perp\!\!\!\perp_H Y|Z[\Pi]$ if $X \perp\!\!\!\perp_M Y|Z[\mathcal{P}]$ and $\theta_{(X,Z)} \perp\!\!\!\perp_P \theta_{(Y,Z)}|\theta_Z[\Pi]$.

Belief Function C.I. (Dempster-Shafer): Let S be a random subset of Ω, and $\tilde{S}_X = \{X(\omega) : \omega \in S\}$. Then $X \perp\!\!\!\perp_{\text{Bel}} Y|Z$ if $X \perp\!\!\!\perp_V Y|Z[S]$ for all $S \subseteq \Omega$ and $\tilde{S}_{XZ} \perp\!\!\!\perp_P \tilde{S}_{YZ}|\tilde{S}_Z$.

3.2 Types of Graphical Model

We can now define various kinds of graphical model using conditional independence. Principally there are three types which have been used to represent relations of probabilistic c.i. between variables: the undirected graph, directed acyclic graph and chain graph.

3.2.1 Undirected graphs

We start with the simplest version, the undirected graph. The definition we use here is the same as that given by Lauritzen et al (1990).

Definition 3.2 *Let $Q = Q_1, \ldots, Q_n$ be a set of random vectors. Then an* undirected graph G_Q *consists of the set nodes $\{Q_i : Q_i \in Q\}$ and a subset of the arcs $\{(Q_i, Q_j) : Q_i, Q_j \in Q, i \neq j\}$, consistent with the* global Markov property:

For any subsets $A, B, C \subseteq Q$,

$$A \perp\!\!\!\perp_P B | C$$

if every path between a node in A and a node in C passes through a node in B.

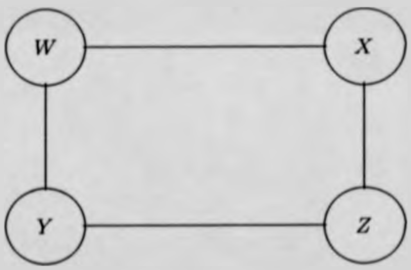


Figure 3.2. An Undirected Graph

For instance the graph in figure 3.2 implies the c.i. relations

$$W \perp\!\!\!\perp_P Z | X, Y \quad \text{and} \quad X \perp\!\!\!\perp_P Y | W, Z \ .$$

Note, however, that the converse of the global Markov property is in general false, so that figure 3.2 does not deny, for example, the relation

$$W \perp\!\!\!\perp_P X | Y, Z \ .$$

Hence adding an arc to an undirected graph does not lead to any untrue implications — it merely reduces the amount of information in the graph.

Furthermore, it should be noted that while sparser graphs (that is, graphs containing fewer arcs) are more informative given any set of c.i. constraints, there is in general no unique minimal graph, and not all sets of c.i. conditions can be represented both simultaneously and exclusively by an undirected graph. For example, the relation $X \perp\!\!\!\perp_P Y$ cannot be represented in an undirected graph G_Q without implying $X \perp\!\!\!\perp_P Y | Z$ for every $Z \in Q \setminus (X \cup Y)$. As Simpson's Paradox illustrates, this is in general false.

Further details on representing c.i. in undirected graphs may be found in Pearl et al (1989) and Whittaker (1990).

3.2.2 Directed acyclic graphs

There appears to be some confusion in terminology in the literature on directed graphs, with some authors using the term 'directed acyclic graph' and some the term 'influence diagram' (some even use both) to define the same object. In this chapter I intend to follow the terminology used by Pearl (1988), namely that a graph representing only probabilistic (random) quantities will be called a directed acyclic graph, or DAG, and one which includes decisions and utilities will be called an influence diagram (ID).

The development of DAGs was largely inspired by the work of Pearl (1986) on propagation within probabilistic networks and the earlier work of Kiiveri et al (1984). While a DAG looks much the same as an undirected graph, excepting that the arcs have arrows on them, there is a subtle difference in the way it is defined and interpreted. We begin with some notation for use in directed graphs.

Consider a graph on the set of objects Q. Let $Q_i, Q_j \in Q$.

$\mathrm{An}(Q_j) := \{Q_i \in Q :$ There exists a directed path $\gamma(Q_i, Q_j)\}$ is the *ancestor set* of Q_j.

$\mathrm{De}(Q_j) := \{Q_i \in Q :$ There exists a directed path $\gamma(Q_j, Q_i)\}$ is the *descendent set* of Q_j.

$\mathrm{Pa}(Q_j) := \{Q_i \in Q :$ There exists a directed arc $(Q_i, Q_j)\}$ is the *parent set* of Q_j.

$\mathrm{Of}(Q_j) := \{Q_i \in Q :$ There exists a directed arc $(Q_j, Q_i)\}$ is the *offspring set* of Q_j.

So, for instance, in figure 3.3, $\mathrm{An}(Z) = \{W, X, Y\}$.

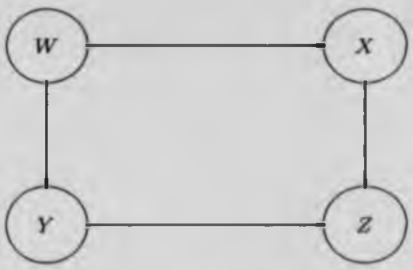


Figure 3.3. A Directed Acyclic Graph

Definition 3.3 *Let $Q = Q_1, \ldots, Q_n$ be a set of random vectors. Then a* directed acyclic graph, *or* DAG *H_Q consists of the set of nodes $\{Q_i : Q_i \in Q\}$ and a subset of the directed arcs $\{(Q_i, Q_j) : Q_i, Q_j \in Q, i \neq j\}$, such that for all $Q_i \in Q$,*

(i) $Q_i \notin \mathrm{An}(Q_i)$, *in other words, H_Q has no cycles, and*

(ii) $Q_i \perp\!\!\!\perp_P Q \setminus (Q_i \cup \mathrm{De}(Q_i)) \mid \mathrm{Pa}(Q_i)$.

The standard definition of a DAG involves specifying an ordering (or at least a partial ordering) of Q, and defining each successive random vector in the ordering as an offspring of some subset of the preceding random vectors. Indeed, this is how a DAG is often constructed; see, for example, Smith (1989b). However, the above definition is equivalent since, without loss of generality, we may assume that $Q_i \in \mathrm{An}(Q_j)$ only if $i < j$.

3.2.3 The d-separation theorem

We defined the undirected graph using the global Markov property, which allows us immediately to read off all c.i. statements implied by the graph, simply by using the separation criterion. The DAG, in contrast is defined much more sparsely (in terms of as few as $n-1$ c.i. statements, to be precise). Yet the number of possible c.i. constraints between n variables is $\binom{n}{2}2^{n-2}$, most of which do not involve conditioning on parental variables, as in the defining conditions.

Fortunately, there is a method of determining which c.i. statements are implied by a DAG, known as d-separation. This first appeared in Pearl and Verma (1987), and was proved by Verma (1988). Geiger and Pearl (1988) proved the closure of the DAG representation of conditional independence, in the sense that the c.i. conditions implied by a DAG are exactly those the verity of which can be derived using d-separation.

The version of d-separation we use here is due to Lauritzen et al (1990). It uses a process of moralisation, as described by Lauritzen and Spiegelhalter (1988), to form an undirected subgraph of the original DAG, which is then consistent with the global Markov property.

Theorem 3.1 (d-Separation) *Suppose that conditional independence statements for a set of random vectors Q are represented in a DAG H, and that $A, B, C \subseteq Q$ denote subsets of random vectors on it. Adapt the DAG in the following way:*

(i) Form the directed subgraph H_1 of H whose nodes consist of the ancestor set $A \cup B \cup C \cup \mathrm{An}(A \cup B \cup C)$ and whose directed arcs are those in H which lie between these nodes.

(ii) For all $Z \in I_1$, join all unconnected pairs of nodes $(X, Y) \in \mathrm{Pa}(Z)$ by an undirected arc. This process is known as moralising the graph since all parents of a single node are joined by an arc. Call this mixed graph H_2.

(iii) Form an undirected graph I by replacing all directed arcs in H_2 by undirected arcs.

Then,

$$A \perp\!\!\!\perp_P B | C$$

if all undirected paths in I between a node $R \in A$ and a node $S \in B$ must pass through a node $T \in C$.

An example of the deduction of a c.i. statement via d-separation is illustrated in figure 3.4.

As with the undirected graph, the DAG has its limitations; it is not always possible to represent a set of c.i. statements faithfully in a single DAG, even if it is representable in an undirected graph. For example, the two c.i. conditions $W \perp\!\!\!\perp_P Z | X, Y$ and $X \perp\!\!\!\perp_P Y | W, Z$ illustrated in figure 3.2 cannot be simultaneously implied by a single DAG. The work of Studený (1992) is one attempt to overcome the problem of incompleteness in both undirected graphs and DAGs.

Nevertheless, as was shown by Pearl and Verma (1987), the DAG can encompass a much wider range of c.i. models than the undirected graph. In particular it is much more capable of expressing causality (see for example Pearl and Verma 1991, Pearl and Wermuth 1992), although directed arcs in a DAG do not necessarily represent causal relationships.

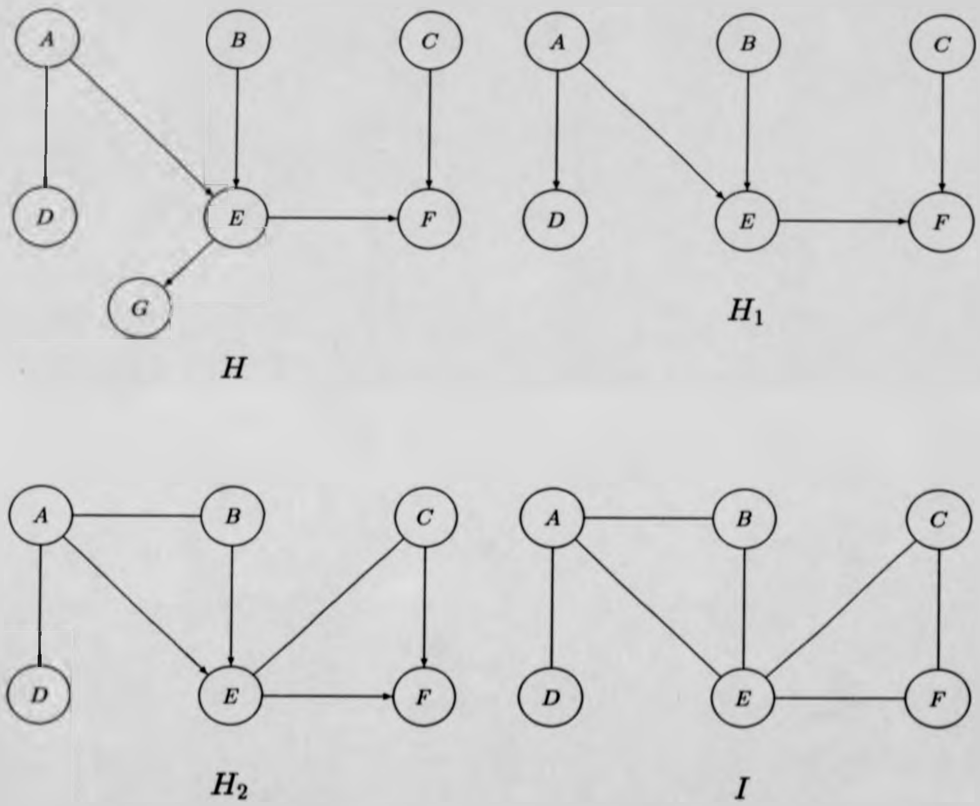


Figure 3.4. Verifying the statement $D \perp\!\!\!\perp_P F | E$ using d-separation

Pearl (1993) gives a review of the development of DAGs over the past decade.

3.2.4 Chain graphs

One way to overcome some of the restrictions with the undirected and DAG forms is to combine both directed and undirected arcs in the same graph, known as a *chain graph*. This idea was first suggested by Verma (1988) and used by Lauritzen and Wermuth (1989). The properties of chain graphs were described by Frydenberg (1990).

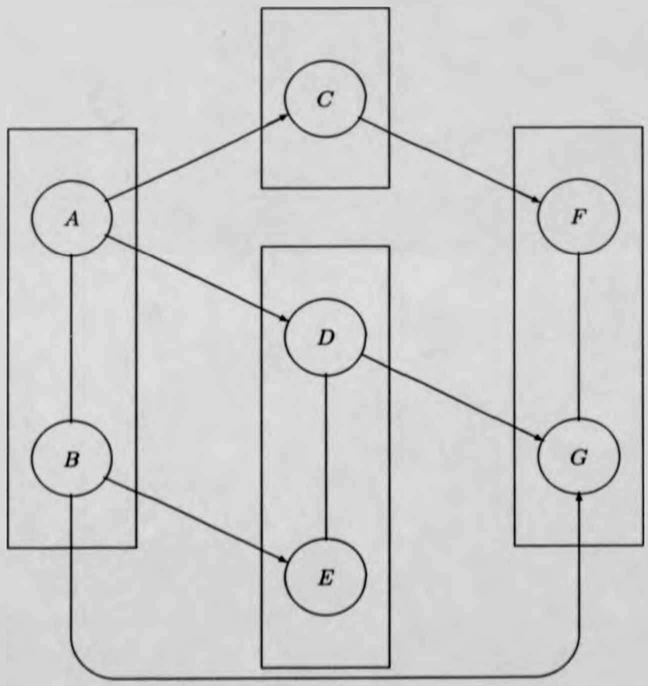


Figure 3.5. A Chain Graph

Clearly such graphs are in a trivial sense more general than either the DAG or undirected graph, but they can also encompass c.i. models which neither of the other two forms are able to illustrate. For example, where the natural ordering of a set of variables is only partial, it may be that there is no obvious reason why an arc between two dependent variables should point one way rather than the other. In such cases, linking them with an undirected arc in a chain graph may result in a more efficient representation of the c.i. structure. Chain graphs are usually drawn in a block-format, as shown in figure 3.5.

As given by Frydenberg (1990), the chain graph is characterised by the following properties:

(i) There are no cycles which both contain a directed element and respect the direction of all arrows.

(ii) The chain Markov property. This is equivalent to the d-separation criterion for DAGs, except that a chain component (that is, a set of nodes all of which are connected to each other by an undirected path) is considered as a single offspring, for the purposes of moralising the graph.

3.2.5 Applications of graphical models

While there have been a number of applications for undirected graphs (Whittaker 1990) and chain graphs (Wermuth and Lauritzen 1990), by far the greatest interest has been shown in DAGs, particularly in conjuction with Bayesian methodology and in the context of expert systems.

The majority of such expert systems are in the field of medical diagnosis. Typically, a provisional model will be drawn up based on the opinions of medical experts (see for example Spiegelhalter and Cowell 1992), including a DAG representing the relations between all the variables perceived to be relevant, together with an 'expert prior' distribution for each variable, conditioned on its parents. An example from Lauritzen and Spiegelhalter (1988) of such a DAG is shown in figure 3.6.

Once a DAG has been constructed, and an appropriate joint prior distribution specified, data can then be used to update the marginal distributions of each variable. This process was first described by Pearl (1986) for the singly-connected DAG (that is where no two nodes have more than one path between them), using the Bayesian paradigm. It was later generalised to all DAGs by Lauritzen and Spiegelhalter (1988), and to the updating of conditional distributions by Spiegelhalter and Lauritzen (1990).

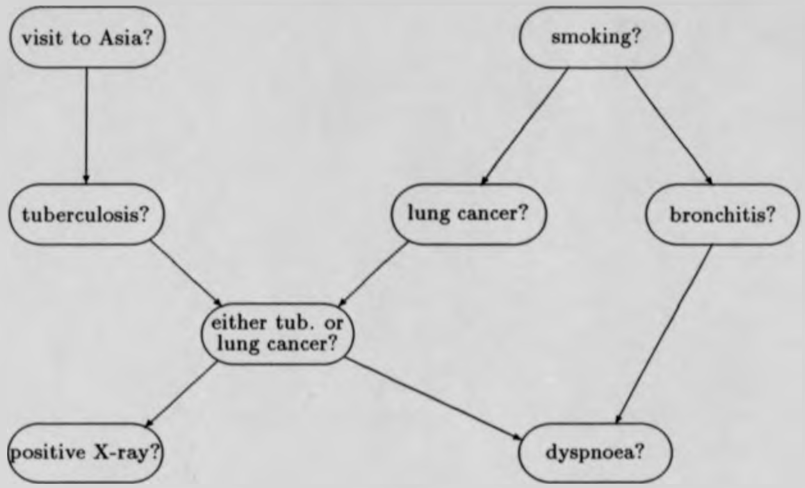


Figure 3.6. An example of a DAG used in medical diagnosis

While most of the work in this area has concentrated on how to refine the model quantitatively using data, there has been increasing interest in the use of data to help construct and modify the qualitative aspects of the model, namely the shape of the DAG itself. Pearl (1988) and Speed (1990) were among the first to consider how data might be used to inform on the structure of a DAG. Cowell et al (1993) consider the use of scoring rules to evaluate how well the model fits the data qualitatively, and Spiegelhalter and Cowell (1992) describe a learning procedure which uses data to refine both qualitative and quantitative aspects of the model.

A review of the latest developments in this area may be found in Spiegelhalter et al (1993).

3.3 The Use of Graphical Models in Decision Theory

The theory of graphical modelling can be extended to encompass models of a decision-theoretic nature. Such models are of particular interest as they take us one step closer to our goal: the modelling of games.

3.3.1 Influence diagrams

An influence diagram is essentially a DAG adapted for use in decision analysis. This involves the introduction of two additional types of node. The chance nodes used in the DAG are still present, represented by circles. Decisions to be taken by the decision maker are represented by squares, and his utility by a diamond. These additions lead to two more conditions an ID must fulfill, over and above those required of a DAG.

Definition 3.4 *Let $R = R_1, \ldots, R_n$ be a set of random vectors, $D = D_1, \ldots, D_n$ be a set of decisions to be taken by the decision maker Π, and U be Π's utility. Let $Q = (Q_i) = R \cup D \cup U$. Then an* influence diagram, *or* ID I_Q *consists of the set of nodes $\{Q_i : Q_i \in Q\}$ and a subset of the directed arcs $\{(Q_i, Q_j) : Q_i, Q_j \in Q, i \neq j\}$, such that,*

(i) for all $Q_i \in Q$, $Q_i \notin \mathrm{An}(Q_i)$, in other words, I_Q has no cycles,

(ii) for all $Q_i \in Q$, $Q_i \perp\!\!\!\perp Q \setminus (Q_i \cup \mathrm{De}(Q_i)) \mid \mathrm{Pa}(Q_i)$,

(iii) the directed arc (Q_i, D_j), where $D_j \in D$, implies Π knows Q_i when taking decision D_j, and

(iv) $\mathrm{De}(U) = \emptyset$.

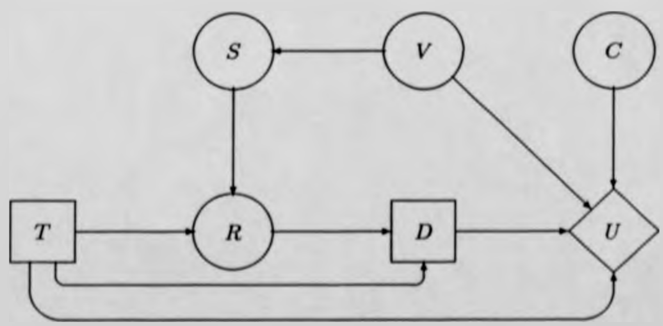


Figure 3.7. An ID Representation of the Oil Wildcatter's Problem

Figure 3.7 shows an ID representation (due to Shachter 1986) of Raiffa's (1968) oil wildcatter problem. The wildcatter first decides whether or not to test the ground (T). He observes the test results R, which are related to the seismic structure S and volume of oil V. Having observed the test results, he decides whether to drill (D), whereupon his utility is a function some or all of T, D, V and C, the cost of drilling.

Two things are worth noting. Firstly, the d-separation criterion applies as in DAGs. However the interpretation is subtly different where decision nodes are involved, so the c.i. relations cannot be said to be strictly of a probabilistic nature. Therefore I will use the non-specific c.i. notation; in the above example we have $T \perp\!\!\!\perp V|S$ and $C \perp\!\!\!\perp V, S, T$.

Secondly, the ID in figure 3.7 displays the property of *perfect recall*, or *'no forgetting'*, as defined by Shachter (1986):

For $D_i, D_j \in D$,

$$D_i \in \text{An}(D_j) \implies D_i \cup \text{Pa}(D_i) \subseteq \text{Pa}(D_j).$$

Shachter (1986) also required that there exist a directed path which passes through every decision node. We will see later what happens when these conditions are relaxed.

The influence diagram was first conceived (by Miller et al 1976) as part of a decision analysis computer system, but it was not until Howard and Matheson (1981) made the connection to probabilistic independence that the potential of the ID as an aid to decision making became apparent. Olmsted (1983) proposed a system for evaluating a decision problem using an ID, and Shachter (1986, 1988) completed the process, giving an algorithm for evaluating directly a wide class of IDs.

A number of rules have been developed for the drawing, manipulation and evaluation of IDs. These include the addition (Smith 1988) and deletion (Olmsted 1983) of nodes, arc reversal (Howard and Matheson 1981) and arc deletion (Smith 1989a), rules for taking advantage of specific types of dependencies between nodes (Smith 1989a, Tatman and Shachter 1990) and an algorithm for using the above operations (Shachter 1986). A more extensive list of rules is given in Smith (1988). Howard (1990) gives a good introduction to IDs and how to draw them in practice.

IDs have a number of uses. They can help with elicitation and the simplification of a problem (Smith 1989a, 1994). They can be used to calculate the expected value of information (Shachter 1986, Matheson 1990), and they can incorporate a variety of systems including belief adjustment (Goldstein 1990) and state-space forecasting (Smith 1990). Oliver and Smith (1990) contains a number of articles on applications of influence diagrams.

3.3.2 A comparison with decision trees

The decision tree has been around for a long time and is a tried and tested graphical aid to decision analysis. So why do we need the ID, which on the face of it appears to involve a more difficult technique for analysing a problem? Well the first issue is one of complexity of the graph. For example the ID of the oil wildcatter's problem in figure 3.7 comprised only seven nodes and nine arcs. The decision tree for the same problem has over 150 branches.

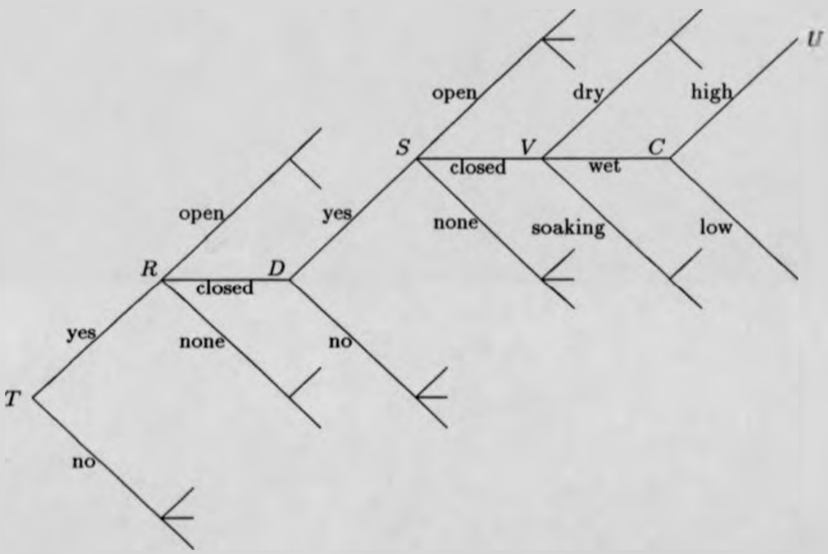


Figure 3.8. A section of the decision tree for the oil wildcatter's problem

While putting in an additional variable or an extra decision adds only one node to an ID, it can double the size of a tree, or worse. However, an ID is also capable of modelling problems with continuous variables or decision spaces, something which can not be done with a tree, except in a schematic fashion. In both cases, the additional information required to solve a decision problem consists of a set of contingency tables, so I would argue that the comparison is a fair one.

The second advantage is computational efficiency. When evaluating a decision problem using a tree, the usual method is to use backward induction, taking expectations or maximising as appropriate. Although this process is in itself quite straightforward, calculating the contingencies at each stage may not be. Such calculations implicitly use conditional independence; in the above example, we have $V \perp\!\!\!\perp T, R, D | S$, so the contingencies for V need only involve S.

But the ID is constructed explicitly to take advantage of such independencies as exist. Hence, as is shown by Pearl and Verma (1987) and Matheson (1990), IDs are a far more powerful tool for analysing and solving many decision problems, and the increasing use of IDs (and DAGs in general) in a variety of real-life situations (see Oliver and Smith 1990, Spiegelhalter et al 1993) bears witness to this claim. As the complexity of a problem increases, the computational advantage of an ID representation over a tree also tends to increase.

The third advantage of the ID and, I believe, the most important is its ability to accurately depict the qualitative structure of the model. The order of variables in a tree is very much constrained by the conditions on what a decision maker knows when taking a particular decision. This can lead to quite a cumbersome structure; in the oil wildcatter's problem, for example, the result of the test of seismic structure is introduced before the seismic structure itself.

While the ID clearly illustrates the knowledge base for each decision, there is much more freedom in how it is drawn. The natural ordering of variables and decisions in a problem is often the best, although slight variations may yield greater computational efficiency. For example, in figure 3.7, the arc (V, S) could if it were more convenient be drawn the other way. Choosing the order in which variables are introduced can be tackled using the theory of dynamic programming; see for instance Kaplan (1982).

This aspect is most useful at the early stages of model building and evaluation (Smith 1994), but it also allows the decision maker to apreciate which variables are most pertinent to their problem, and which decisions are likely to be the most important. The ID may give a better clue as to *why* a particular decision might turn out to be optimal, in contrast to the handle-turning backwards induction approach based on the tree.

There are some problems which a tree models better than an ID. These tend to be either extremely simple cases (which comprise most of the examples of decision trees in the literature) or cases with very little symmetry, in other words where each branch from a vertex leads to a different set of possible values for the subsequent decision or random variable. Fung and Shachter (1990) have attempted to overcome these shortcomings by creating a tree-ID hybrid.

Nevertheless, I would argue that many decisions taken in real life are sufficiently complex and symmetrical to severely reduce the efficacy of trees in analysing them.

3.3.3 Other graph-based systems

Before the advent of the influence diagram and the algebra of conditional independence, much of the use of graphical and causal models was non-probabilistic or even non-quantitative, with systems such as non-monotonic or fuzzy logic much in evidence (see Blalock 1971). In the last decade, the use of probability has taken over. In fact this has been almost exclusively of a Bayesian nature, the reason being that anything other than a completely connected graph gives rise to a hierarchical model, for which the superiority of Bayesian methodology over classical is widely acknowledged.

However, for those who are unwilling to specify a point probability for an uncertain event, or for whom the problems associated with vague prior distributions lead to a feeling of uneasiness, there is a non-Bayesian approach which is currently in use. It is based on the Dempster-Shafer belief function, a type of lower probability; see Dempster (1968) and Shafer (1976, 1981) for details on the foundations of the belief function methodology.

The belief function is used by Dempster and Kong (1988) in conjunction with a graphoid structure based on logical relations. Dempster (1990) extends this structure to combine both logical and probabilistic relations between variables. Shafer and Shenoy (1988) have shown how belief functions can be propagated through an undirected graph known as a Markov tree.

Slightly closer in nature to the ID is Shenoy's (1990) valuation network, an example of which for the oil wildcatter's problem is shown in figure 3.9. The utility is split into two parts: profit from the test κ and profit from drilling π. There is now an explicit prior probability ρ represented in the network, and the conditional probability μ plays the same role as the unobserved variable S in the original version. The cost of drilling is not explicitly included (but could be if necessary).

The main difference to the other graphical models described in this chapter is that the interpretation of the directed and undirected arcs is somewhat different. The system is geared to calculation of the conditional probabilities, rather than to describing the problem from a decision maker's point of view. Thus, although Shenoy (1990) claims that the computations involved are more efficient than those based on an ID representation, the system is much harder to understand, so some of the modelling benefits of the ID are lost.

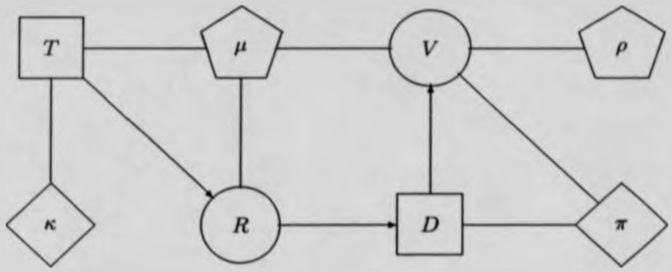


Figure 3.9. Shenoy's (1990) Valuation Network of the Oil Wilcatter's Problem

Shenoy (1991) shows how Dempster-Shafer belief functions, Spohn's (epistemic) disbelief functions and Zadeh's possibility functions fit into the framework of a valuation-based system.

3.3.4 More than one decision maker: towards a theory of games

Consider again the oil wildcatter's problem, and recall that the ID shown in figure 3.7 conforms to the axiom of perfect recall, or Shachter's (1986) 'no forgetting' rule. In fact it makes no difference in this case, since $\mathrm{Pa}(T) = \emptyset$ and T is completely determined by R — in other words, if you know the (possibly null) result of a test, you will also know whether the test took place.

But in general this will not be so. The original description of the oil wildcatter's problem by Raiffa (1968) included the possibility that he would have some background knowledge K_T (on the geography of the surrounding area and the type of seismic structure and volume of oil to expect). Now suppose the decisions are made by a company rather than an individual, and that the decisions on whether or not to test and drill are made by two different people within the company. Typically the driller will have some subset of the tester's background information, which together with some information on the likely costs of drilling forms his knowledge base K_D. This can be modelled by the ID in figure 3.10.

Notice that the driller does not have all the available relevant information at his disposal. The tester does not pass on her entire background information, which might include specialist knowledge and past experience. It is still, however, a single decision making process, in which the decision maker (the company) 'forgets' some information between one decision and the next. We say the decisions are taken by two individual agents, but with the same goal or utility.

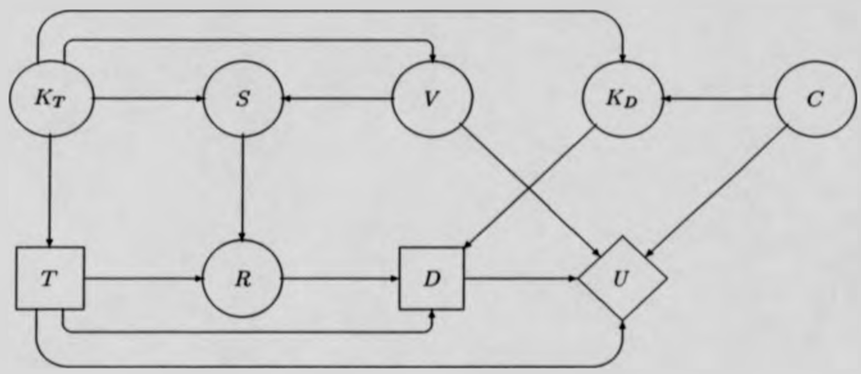


Figure 3.10. An ID of the Oil Company's Problem

Now let us inject a further slice of realism into the model, by considering the personal objectives of the two individuals involved. To illustrate the point we will assume that they are only concerned with their own standing within the company, and don't care one way or the other about the profit made by the company on this particular venture.

First the tester who needs to show that she knows when to test for seismic structure. Her utility U_T will depend on the decision to test T, whether her results R give a good indication of the volume of oil V, and in terms of professional pride on whether R matches the true seismic structure S. The opportunity to observe V and S will depend on the decision to drill D. Finally recall that R determines T, so we have $\text{Pa}(U_T) = \{V, S, R, D\}$.

As for the driller, his utility U_D will depend on the volume of oil V and the cost of drilling C, both of which will be observed depending on D. So $\text{Pa}(U_D) = \{V, C, D\}$. The complete model is shown in figure 3.11.

This is no longer a simple decision problem. There are three decision-makers involved: the tester, the driller and the company. The company does not have a decision to make and is passive. (It may decide to promote or dismiss either of the other two, but that is part of a much larger process, outside the scope of this model.) The decision problem has now become a game.

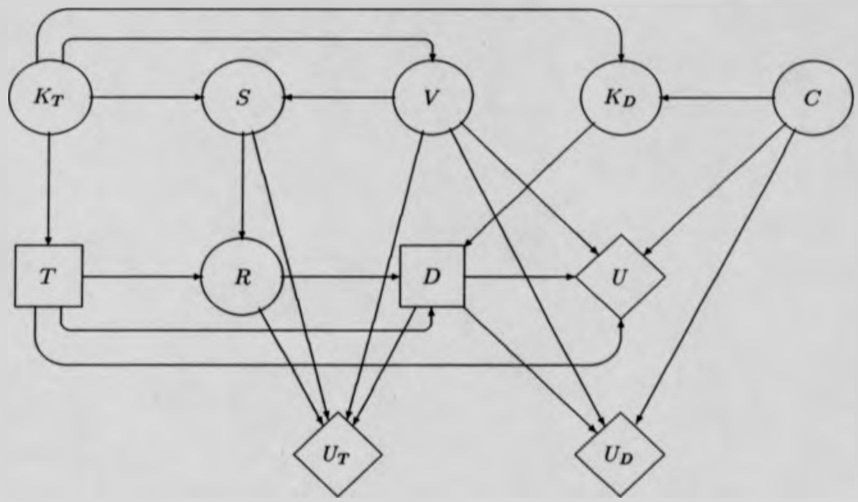


Figure 3.11. Even more problems for the Oil Company!

Although the extension of the model to encompass games looks fairly innocuous, there are some major issues to be considered. Firstly, each actor will view the actions of the other actors as variables, determined not by any chance mechanism (to which the well-founded calculus of probability can be applied) but by the thinking processes (rational or otherwise) of the other actors with respect to *their* utility.

This consideration is tackled by Smith (1988, 1994) and by Smith and Allard (1992). However, the presence of more than one actor raises even more profound questions. What do the actors know about each other? What factors will other actors take into account when taking their actions? How can a actor learn more about the other actors, even about himself? Does every actor have the same perception of the game, of even of the influence diagram used to model it?

A new foundational framework is required to tackle these issues; as far as I am aware, the only attempt made so far to build such a framework is that of Allard and Smith (1992). In the next chapter I seek to improve and extend this foundational framework.

4 The Basic Model

4.1 Preliminaries

As we have seen, extending the use of influence diagrams from the decision problem involving a single decision maker to a game with two or more actors is not straightforward. It requires that we make a whole new set of assumptions about the nature and motivation of the actors involved. Now, I aim to state explicitly and justify each assumption being made, which means going right back to the foundations of game theory.

4.1.1 The definition of a game

In this chapter we construct, step by step, a new framework for modelling games based on the influence diagram representation. We start with a definition of a game which is stated in enough generality so as to encompass a very wide class of game-type models. It is consistent with other definitions employed in the game theory literature.

Definition 4.1 *A* game Γ_A *is played by:*

(i) a finite set of sentient actors $\Pi_S = \{\Pi_1, \ldots, \Pi_m\}$, *and*

(ii) the actor nature Π_N.

$\Pi = \Pi_S \cup \Pi_N$ *is the set of actors.*

Γ_A *consists of a finite set of* actions $A = \{A_1, \ldots, A_n\}$. *Each action* $A_i \in A$ *is taken by the corresponding* agent $\pi_i \in \pi = \{\pi_1, \ldots, \pi_n\}$, *where* π_i belongs to *an actor* Π_j *(*$\pi_i \in \Pi_j$*) for some* $\Pi_j \in \Pi$. *We say that an agent which belongs to a sentient actor is sentient.*

We also require the following terms.

Definition 4.2 *If an action* A_i *is taken, it assumes a* value $A_i = a_i$, *where* $a_i \in \mathcal{A}_i$ *is the* act *chosen by* π_i, *and* $\mathcal{A}_i$ *is the* action space *for* A_i.

For any subset of actions $X = \{A_j\} \subseteq A$, *we define the set of acts (or values)* $x := \{a_j : A_j \in X\}$ *and the action space* $\mathcal{A}_X := \bigotimes_{A_j \in X} \mathcal{A}_j$.

There are a number of points to note here. The first is that we have used the symbol $\in$ to denote two different relations: an agent belonging to a set of agents, and an agent belonging to an actor. The difference in meaning is related to the philosophical question of distinction between an actor and his set of agents. However, that need not concern us for the time being since identifying the two makes no difference to our model. Indeed we could just as well define an actor to be a subset of agents, with Π partitioning π.

It is also worth considering the definition of nature as an actor. All the events in a game determined randomly or by chance are attributed to actions taken by nature. This device has been used since the start of game theory (see for example Kuhn 1953, Blackwell and Girshick 1954). The 'referee' employed by von Neumann and Morgenstern (1947) performs the same role.

There are two reasons why we use this definition. The first is purely for notational convenience. The second is to emphasise the similarity between the actions taken by sentient agents and those taken by agents of nature. Even though different types of process are involved, namely the conscious decision and the chance or random process, they both result in the same type of object, namely an act. And once an act is chosen, and the value of that action determined, there is essentially no difference in quality between the sentient act and the act of nature. Only when we start to consider restrictions on the behaviour of sentient agents and make assumptions about the rationality of actors does the distinction between the types of action become apparent.

The device of the 'agent', as used, for example, by Kuhn (1953) and Harsanyi and Selten (1988) is also well known in game theory. We will base our theory on the agent rather than the actor primarily for reasons of generality. Initially, the entity of the actor will not enter our consideration except to motivate the definitions. In particular, we do not at this point assume any special connection between the actions taken by agents belonging to the same actor. Hence in a restricted sense, the number of actors in a game makes no difference to the qualitative structure we use to describe the game.

4.1.2 Some remarks on the definition

I would like to comment on the generality of our definition of a game, and emphasise the paucity of assumptions made. As defined above, the term game covers just about any situation which involves one or more people doing things. There is no presumption as to why or how or when an action is taken. Conditions such as rationality, perfect recall and common knowledge have not been imposed; the implications of such conditions will be examined later.

There is, however, one important assumption we make, namely that the game is finite. Thus immediately we rule out a large class of games which have historically played a significant role in game theory. Now the inability to represent such an important genre of games might be considered a serious deficiency in any modelling framework. And I would not deny the possibility that the ID representation could be extended to include infinite games, in the same way that a DAG can be used (see Smith, 1990) to model an infinite time series. However there are good reasons why I choose not to tackle them here.

There are many ways of classifying games, whether it be by number of actors, perfectness or completeness of information, one-shot or repeated, finite or infinite games. And both the extensive and normal form models of games have different strengths and weaknesses when it comes to representing these different classes of games. The influence diagram model which I define in this chapter is equally capable of representing games with imperfect or incomplete information, the number of actors makes little difference, and finitely repeated games can be accommodated without too much difficulty.

In fact, action spaces with uncountable cardinality can be accommodated just as easily as the countable or finite. But there are serious issues involved if we wish to consider games with infinitely many actions. For a start it is well known that an infinite game cannot simply be considered as the limiting case of a finite game, hence the well-documented divergence in results; see Howard (1971). Furthermore there are foundational problems which present themselves; for example, the extension of a theorem from a space with finitely many to countably many dimensions is non-trivial. In the particular case of the ID, while the (weaker) local Markov property may apply equally to an infinite graph as to a finite one, the extension of the global Markov property is not so straightforward.

Intuitively, we may consider infinite games to be on a completely different level *in a qualitative sense* from finite games. And since the ID representation emphasises especially the qualitative aspects of a game, we might expect that the foundational treatment of the infinite would need to be quite different from that of the finite. Detailed consideration of the foundations of infinite games is beyond the scope of this thesis.

4.1.3 Scheduling graphs

We will now begin to introduce some basic assumptions regarding the qualitative structure of the games which are to be considered. These assumptions are common to every framework for modelling games of which I am aware: so common that they are rarely stated. Nevertheless in the spirit of comprehensiveness, which I propounded above, each assumption made will be stated explicitly.

Definition 4.3 *We say an agent π_j* admits *A_i if the value $A_i = a_i$ is an input to the process of taking the action A_j.*

Let us consider how this should be interpreted. Firstly, there is the assumption of a process, or *algorithm* in computing terms, which determines which act $a_i \in \mathcal{A}_i$ is to be chosen. What does it imply for a particular value to be an input of such an algorithm? If π_j is sentient, then we would say that π_j knows or has observed $A_i = a_i$ when it takes action A_j. In this respect we consider the faculties of an actor to be delegated to his agents.

In the case where π_j is an agent of nature, A_j could be thought of as a function of A_i, or more generally as a response to A_i. Such functions can be either deterministic or stochastic. For example, in the context of a probabilistic system, A_j would be distributed with some set of parameters which include A_i.

Definition 4.4 *A* scheduling graph *G is a directed acyclic graph on A such that $A_i \in \mathrm{Pa}(A_j)$ if and only if π_j admits A_i.*

A game Γ_A is scheduled *if its scheduling graph exists.*

To ensure the existence of a scheduling graph, we require that the pattern of admissions does not induce any cycles, in other words that,

> there is no sequence $(A_{(i)})_{i=1}^{m} \subseteq A$ such that $\pi_{(1)}$ admits $A_{(m)}$ and $\pi_{(i+1)}$ admits $A_{(i)}$ for $i = 1, \ldots, m-1$.

Thus we rule out games involving clairvoyance and similar phenomena.

In addition, it must be clear whether or not π_j admits A_i for all pairs of actions (A_i, A_j). This is not a trivial assumption, since it rules out games where the order of some of the actions is undetermined, and may depend on another action taken during the game. An example might be a game of chess where the first action A_0 is the toss of a coin to determine who plays white.

However, this restriction does not necessarily prevent us from modelling such games; if the rules of a game allow for the order of some actions to be determined by the value of another, we can usually overcome this by carefully defining the actions, and if necessary including additional actions some of which may turn out to be null, depending on another value in the game. So for example in the game of chess, we could add the variable A_1' which returns the value a_1 (the first actor's first move) if A_0 = 'heads', and $\emptyset$ otherwise. (Alternatively we might swap the actors round or analyse the two versions separately).

Note that if it exists, the scheduling graph is unique.

In a scheduling graph, we use the obvious notation: an action taken by an agent of nature is denoted by a circle, one by a sentient agent denoted by a square. A simple example of a scheduling graph is given in figure 4.1. An agent of nature π_1 takes action A_1, choosing an act a_1 which is observed by the two sentient agents π_2 and π_3, as specified by the presence of the directed arcs (A_1, A_2) and (A_1, A_3). Then π_2 takes its action A_2, the result of which is observed by π_3 (according to the arc (A_2, A_3)) before it takes action A_3.

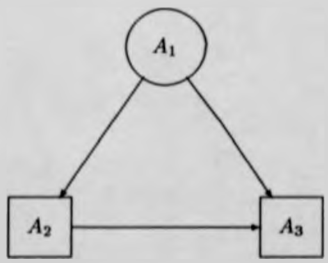


Figure 4.1. A simple scheduling graph

The scheduling graph induces the natural partial ordering on A,

$$A_i \prec A_j \iff A_i \in \mathrm{An}(A_j).$$

Thus an alternative way to define the scheduling graph would be to invoke the (sufficient) partial ordering axioms,

(i) Transitivity: $A_i \prec A_j$ and $A_j \prec A_k \implies A_i \prec A_k$.

(ii) Mutual exclusiveness: $A_i \not\prec A_j$ or $A_j \not\prec A_i$.

This natural partial ordering motivates the following definition.

Definition 4.5 *In a scheduled game, A_i occurs* before *A_j (in* game time*) if $A_i \in \mathrm{An}(A_j)$ in G.*

So for instance, in figure 2.1, A_1 occurs before A_2 and A_3.

There is an important distinction to be drawn between real time and game time. What matters in a game is not when an action is taken, but when the result of that action starts to affect other actions in the game. Thus two games in which the real time order of actions differs, but which nevertheless have the same game time partial ordering, are considered identical for the purposes of our theory. For example, if an action taken by an agent of nature (say the determination of some unknown global constant) is admitted for the first time part of the way through a game, then it makes no difference whether the constant was determined at the start of the game, or immediately previous to its admission. Thus in modelling a game we can choose which representation is most convenient, according to dynamic programming principles as mentioned in the previous chapter.

4.2 Orientation

We now turn to what is perhaps the focal point of game theory: what motivates an agent to choose one act rather than another. We need to consider the effect a game has on an agent and what part its action has to play. In other words, what rewards are available to the agent, what are the preferences of the agent, and how do they affect the way it acts. As always, we start with the basics.

Definition 4.6 *The* outcome *$A = a$ of a game Γ_A is the set of acts $a = (a_1, \ldots, a_n) \in \mathcal{A}$ taken by the agents π, where $\mathcal{A} := \bigotimes_{A_j \in A} \mathcal{A}_j$ is the* outcome space.

4.2.1 Predisposition and utility

Every sentient actor derives some benefit (or loss) as a direct consequence of any outcome. We can describe the relation between outcome and benefit in terms of a predisposition with which each agent belonging to that actor is presumed to be endowed.

Definition 4.7 *The* predisposition V_i *of a sentient agent* $\pi_i \in \Pi_j$ *defines an ordering of the space of outcomes* $\mathcal{A}$ *such that for all* $a, a' \in \mathcal{A}$,

$$V_i(a) \succ V_i(a') \iff a \text{ is more beneficial to } \Pi_j \text{ than } a',$$

$$V_i(a) \sim V_i(a') \iff a \text{ and } a' \text{ are equally beneficial to } \Pi_j,$$

and

$$\pi_h, \pi_i \in \Pi_j \implies V_h = V_i.$$

We define the set $V := \{V_1, \ldots, V_n\}$.

We can then determine the utility to each agent of a particular outcome.

Definition 4.8 *The* utility $U_i(A)$ *of a sentient agent* π_i *is, without loss of generality, a real-valued function satisfying,*

$$U_i(a) > U_i(a') \iff V_i(a) \succ V_i(a'),$$

$$U_i(a) = U_i(a') \iff V_i(a) \sim V_i(a'),$$

$$U_i(a) = U_j(a) \quad \text{if} \quad \pi_i \text{ and } \pi_j \text{ belong to the same actor,}$$

for all $a, a' \in \mathcal{A}$.

We define the set $U := \{U_1, \ldots, U_n\}$

Note that, according to the above definition, the utility may be either cardinal or ordinal.

At first sight it seems as if predisposition and utility are one and the same thing. However, within our concept of a game, they perform distinct roles.

Predisposition can be considered as an expression of the state of an agent at the start of the game, defined in terms of the potential benefits of that game. By definition, it determines a scale by which the gains or losses of the agent resulting from any outcome can be measured. In other words, it defines what is better for the agent. The agent may know its predisposition entirely, or alternatively may have some uncertainty about it. We make no assumption either way.

We might ask how an agent comes to be in a certain state, to have one predisposition rather than another. There are a number of possibilities. The agent may simply be endowed with its predisposition, much in the same way as the hypothetical Bayesian is assumed to be endowed with a prior distribution over some unknown variable. Alternatively the predisposition could be thought of as a function of some other act of nature. An example of this which is quite common in the game theory literature (see for example Harsanyi 1967) is when an actor is presumed to be drawn at random from some population for which the distribution of predispositions is known.

I suppose it is possible that we might want to model a game in which an actor 'joins in half-way through', so that a previous action by a sentient agent might affect the identity of that actor and hence the predisposition of his agents. Nevertheless, in each case we may without loss of generality consider a predisposition V_i as an action taken by an agent of nature $\pi(V_i)$, and we interpret its relations to other actions in the game accordingly. It is, however, a very special type of action, being identified with a specific sentient agent, which is why we have given it separate treatment in the development of our theory.

The utility represents the realisation by an agent of the benefit which results from a particular outcome. The relationship between predisposition and utility is analogous to that between the preposterior and posterior distribution in Bayesian analysis. The importance of distinguishing between them will becomes clear when we incorporate utility into our model, using the orientation graph.

At this point it is worth noting that, although the minimalist definitions of predisposition and utility given above imply the existence only of ordinal utility, other forms of utility can be used within this framework. For example, we could construct a cardinal utility by extending the predisposition to be an ordering of all lotteries over $\mathcal{A}$, as demonstrated by Savage (1954).

4.2.2 The orientation graph

Definition 4.9 *An* orientation graph *H is a directed acyclic graph on $Q = A \cup U \cup W$ where $W \subseteq V$ and:*

(i) the subgraph of H on A is the scheduling graph G, that is $H|_A = G$.

(ii) for all nodes $V_i \in W, Q_j \in A \cup W \setminus V_i$,
$V_i \in \mathrm{Pa}(Q_j)$ if and only if $\pi(Q_j)$ admits V_i and
$Q_j \in \mathrm{Pa}(V_i)$ if and only if $\pi(V_i)$ admits Q_j.

(iii) the utility nodes U are terminal nodes, that is $\mathrm{De}(U) = \emptyset$.

(iv) for all $a, a' \in \mathcal{A}_{A \cup W}$,

$$a|_{\mathrm{Pa}(U_i)} = a'|_{\mathrm{Pa}(U_i)} \implies U_i(a) = U_i(a').$$

(v) H is minimal, that is there is no proper subgraph of H on $A \cup U \cup W$ which satisfies (i)–(iv) above.

From now on we will consider that a game Γ_A consists of the set of objects Q, as defined above.

Given a scheduling graph, an orientation graph will always exist, since putting $\mathrm{Pa}(U_i) = A$ for all i trivially satisfies condition (iv). However uniqueness is not guaranteed, even after fixing the subset W. For example, if A_j and A_k are deterministically related actions taken by agents of nature such that $a_k = a_j + 1$, then A_j and A_k will be interchangeable with respect to membership of $\mathrm{Pa}(U_i)$.

It may seem just a little odd that the definition of the orientation graph does not specify any particular connection between the predisposition and utility nodes of a sentient agent. This surprising omission is deliberate. My intention is to encompass the widest possible variety of games within the modelling framework; to specify a particular graphical connection within the definition would either be over restrictive or long-winded and cumbersome. Thus, while I claim that most 'sensible' games can be modelled using this framework, not all models which fit the framework necessarily correspond to a game which could be described as 'sensible'.

My idea of a sensible game involves some connection between predisposition and utility. I would also like to prevent agents from choosing their own predisposition. While this interesting possibility is allowable given the above definition, it does raise the problem of what motivates an agent to choose a particular predisposition; might we need a pre-predisposition? To achieve these ends, I suggest the following conditions:

(vi) If $V_i \in W$ then $V_i \in \mathrm{An}(U_i)$, and

(vii) $A_i \notin \mathrm{An}(V_i)$.

In fact the equivalent definition in Allard and Smith (1992) specifies $V_i \in \mathrm{Pa}(U_i)$, and I suspect that in many games this will be the case. Allard and Smith (1992) also restrict the orientation graph to the case $W = V$. By allowing ourselves more flexibility in the way a game is represented, we are presented with an additional problem: which predisposition nodes to include. In fact there are no hard and fast rules on this one; it comes down to a matter of judgement as to what aspects of a game we are trying to model. We will return to this question later in the thesis.

Figure 4.2 shows two possible orientation graphs for the game corresponding to the scheduling graph illustrated in figure 4.1. In H_1, the predispositions are not included, and the two utility nodes are drawn to represent each utility being a function only of actions A_2 and A_3. In H_2, the action A_1 represents the equivalent of a 'common prior' (as described by Harsanyi 1967). Each sentient agent knows only its own predisposition, which is in the parental set of its utility.

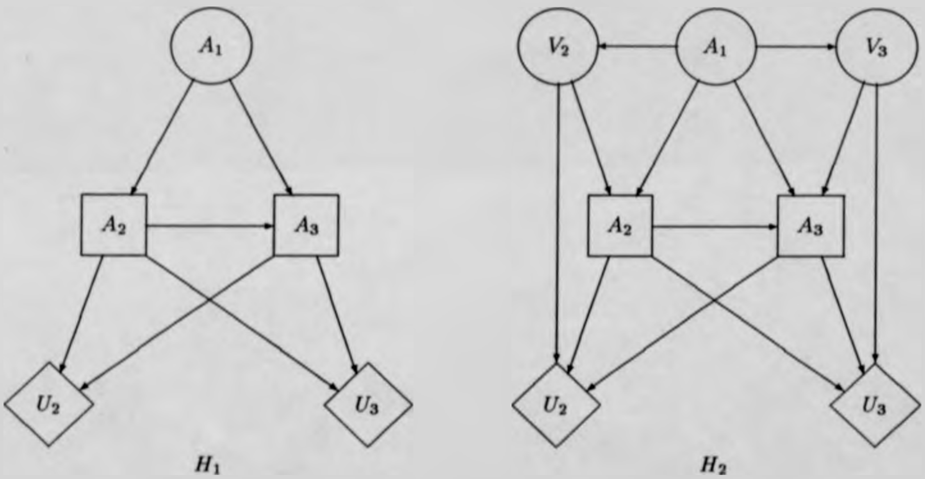


Figure 4.2. Two possible versions of the orientation graph

Now our model is taking shape; it actually looks like an influence diagram. In fact, we have all the ingredients needed to model a game with a single actor possessing perfect recall (the standard requirement in a decision problem). However, we have not yet begun to tackle the issues raised by the inclusion of more than one actor, or even of a single actor with unco-ordinated agents. We now start down that road by making what is probably the biggest assumption in this thesis.

Definition 4.10 *Say a game is* orientated *if its orientation graph H is common knowledge among the sentient agents $\{\pi_i \in \Pi_j : \Pi_j \in \Pi_S\}$.*

Thus for a game to be orientated, we require that every agent has an identical perception of that part of the structure of the game which is prescribed by the orientation graph. Moreover, each agent's perception must include the knowledge that every other agent has an identical perception to itself. This is a very strong assumption. It may hold for simple 'parlour' games, such as chess. But for many 'real-life' situations, it may be implausible as a descriptive assumption.

On the other hand, we have to begin somewhere, and common knowledge of the basic structure of a game is a natural starting point. Indeed most other modelling systems for games require far stronger assumptions involving quite detailed common knowledge of the model. (One exception is Bennett's (1987) theory of hypergames, which allows actors to have differing perceptions of games at the structural level.) As with any other model, ours represents an approximation of the true situation in which certain assumptions are made which may not be correct. It may be interesting to consider the consequences of weakening this common knowledge condition; such considerations are, however, beyond the scope of this thesis.

4.3 Prospective Functions

When seeking to model how an agent may act in a given situation, we assume that the act chosen will depend on the stimuli with which the agent is presented when it takes the action. There are two types of stimuli which may have some effect on an agent's action, and which we therefore need to consider. The first is admission of the values of some of the other actions in the game as defined in the orientation graph. This is sufficient for an action taken by an agent of nature; such an action is determined (up to a stochastic level) by its parental actions.

For an action A_i taken by the sentient agent π_i, however, there is an extra ingredient to take into account. In addition to knowing the set of values $\{q_j : Q_j \in \mathrm{Pa}(A_i)\}$, π_i may have certain beliefs about what values of the set $\{Q_j : Q_j \in Q \setminus (A_i \cup \mathrm{Pa}(A_i))\}$ do or will obtain. Clearly such beliefs may impinge on the decision process of the agent π_i. We choose to model these beliefs via a form of conditional possibility function, known as a prospective function.

4.3.1 Conditioning on experience

Firstly, we need to generalise the admission of an action by a sentient agent to include all hypothetical cases.

Definition 4.11 *Let π_i be a sentient agent. π_i's* experience *Y is defined to be the (possibly hypothetical) admission by π_i of the set $Y \subseteq Q$.*

Definition 4.12 *We define the* prospective function *of a sentient agent π_i to be $B_i(\cdot|\cdot)$ where, for $X, Y \subseteq Q$ and $\mathcal{X} \subseteq \mathcal{A}_X$, $B_i(\mathcal{X}|Y)$ represents π_i's* belief *in $\mathcal{X}$ given experience Y, and*

$$B_i : (\mathcal{X}, y) \mapsto [0, 1].$$

If $B_i(\mathcal{X}|Z) > B_i(\mathcal{Y}|Z)$ then we say that, given experience Z, π_i has a greater belief in $\mathcal{X}$ than in $\mathcal{Y}$.

In principle, we could have defined the prospective function as $B_i(\mathcal{X}|\mathcal{Y})$, where $\mathcal{Y} \subseteq \mathcal{A}_Y$, on all pairs of subsets of subspaces of $\mathcal{A}_Q$. However, according to the definition of admission (and hence of experience), an agent either knows a value or it doesn't. So any subset of an action space other than a single value is inadmissible in that sense. In fact, for any given agent π_i, the only prospective functions which directly impinge on its action A_i are those of the form,

$$B_i(\mathcal{X}|y)$$

for $y \in \mathcal{A}_{\mathrm{Pa}(A_i)}$ and $\mathcal{X} \in \{\mathcal{X} \subseteq \mathcal{A}_X : X \subseteq Q\}$.

4.3.2 Properties of the prospective function

We now consider a number of properties which a prospective function B_i may have.

Let $X, Y, Z \subseteq Q$, with $x, x' \in \mathcal{A}_X$, $y \in \mathcal{A}_Y$ and $z \in \mathcal{A}_Z$.

B1 $B_i(\mathcal{A}_X|y) = 1$.

B2 $B_i(\emptyset[\subseteq \mathcal{A}_X]|y) = 0$.

B3 $B_i(x|x, y) = 1$.

B4 $B_i(x|x', y) = 0$ if $x \neq x'$.

B5 $B_i(x|y) \leq B_i(\mathcal{X}|y)$ if $x \in \mathcal{X} \subseteq \mathcal{A}_X$.

B6 $B_i(x, y|z) \leq B_i(x|z)$.

B7 $B_i(x|y) = 0 \implies B_i(x|y, z) = 0$.

B8 $B_i(x|y) = 1 \implies B_i(x|y, z) = 1$.

B9 $B_i(x, y|z) = B_i(x|y, z) \cdot B_i(y|z)$.

The first issue which arises is whether a term $B_i(\cdot|y)$ is well defined for a given experience Y. For instance, this might not be the case when $B_i(y|\emptyset) = 0$. Since properties B1 and B2 relate to whether an agent believes the set of actions X will be taken at all, we will consider them to be axiomatic, thus defining the terms involved. For every other property, we need to add the caveat: 'provided every term is well defined'.

Next, we consider properties B3 and B4. In order to gain any useful insight, we have to assume that admitting a fact, having observed it or knowing it, an agent will believe that it is true; as the saying goes, 'seeing is believing'. Conversely, an agent should believe false anything which is (logically) mutually exclusive to anything it admits. B3 and B4 represent these principles in the context of a prospective function, including one for which $B_i(\cdot|\cdot) = 1$ implies certainty. This motivates our next definition.

Definition 4.13 *An agent is said to be* intelligent *if B3 and B4 hold provided every term is well defined.*

This is our first step along the road towards a theory of rationality. In principle, it is quite a big step since it involves putting restrictions on the beliefs an agent may hold; as we have seen, this is a highly contentious area within game theory. Nevertheless, it does seem reasonable to require that any data which forms part of the input to a thought process is preserved (in some form) in the output.

Properties B5–B8 are monotonicity conditions. Property B9 represents a further restriction on the shape of an agent's prospective function. Such a condition applies to a number of prospective functions, although in some cases the equality may be replaced by an inequality. The list of properties given above is by no means exhaustive. B3–B9 can be extended naturally to cover beliefs about subsets in the form $B_i(\mathcal{X}|\cdot)$.

4.3.3 Examples

To illustrate what exactly is meant by a prospective function, we now consider a variety of types of belief which an agent in a game might hold. In each case we will assume that the agent is intelligent, so properties B1–B4 hold.

We start with what is perhaps the simplest non-trivial prospective function. The zero-one possibility function (not to be confused with the possibility function of nonmonotonic logic, as described by Zadeh (1979) and others) represents whether an agent believes a certain set of values is possible given some experience.

Definition 4.14 *We say B_i is a zero-one possibility function if,*

$$B_i(\mathcal{X}|y) = \begin{cases} 1 & \textit{if } \pi_i \textit{ believes } X = x \textit{ is possible for some } x \in \mathcal{X} \textit{ given experience } Y = y, \textit{ and} \\ 0 & \textit{otherwise.} \end{cases}$$

Of the properties given above, B5 follows from the definition of the possibility function. B6 and B9 rely on certain regularity conditions associated with coherence of belief. B7 is a corrollary of B6 and B9, and B8 is in general false.

Alternatively, $B_i(\mathcal{X}|Y)$ may express some *degree* of belief about $\mathcal{X}$ given Y on the part of π_i. This might be in terms of upper and lower probabilities, as defined in a subjective framework by Smith (1961), and advocated by Binmore (1991) as an alternative to precise 'Bayesian' probabilities in a game-theory context.

An upper probability may be seen as a refinement of the zero-one possibility function defined above. The use of lower probabilities is far more common, mainly in the form of the Dempster-Shafer belief function. (In fact, while the belief function is formulated as a lower probability, the upper probability of a set is simply the lower probability of its complement)

We denote the conditional belief function $\mathrm{Bel}_i(\cdot|\cdot)$ (as a form of prospective function) and define it according to Shafer (1976). If we assume Dempster's (1968) rule of combination, then B5, B6 and B8 follow immediately. Instead of B9 we have,

B9′ $B_i(x, y|z) \geq B_i(x|y, z) \cdot B_i(y|z)$,

and B7 is in general false.

As we have already seen, belief functions can be incorporated within modelling frameworks based on graphical models. For an overview of some other approaches to probabilistic reasoning, for example the use of previsions, see Fine (1973) and Walley (1991).

At the most structured end, let π_i be an idealised Bayesian, endowed with universal priors and infinite computational capability. Then (provided all terms are well-defined),

$$B_i(\mathcal{X}|Y) = Pr(x \in \mathcal{X}|Y), \text{ for all } \mathcal{X} \subseteq \mathcal{A}_X .$$

Properties B5–B9 all hold.

It is worth noting that even this last special case of the prospective function is more general than the usual Bayesian paradigm, since it makes no assumptions about what forms of belief may be held by other agents, even those belonging to the same actor. Thus it is perfectly possible within our framework to model a game in which, for example, one agent is a Bayesian, another will commit itself only to non-trivial belief functions, and a third refuses to contemplate anything beyond the notion of possibility.

4.3.4 The scope of the prospective function

Earlier, we stated that the predipoosition of a sentient agent could be considered as an action taken by an agent of nature. Consequently, an agent's prospective function will incorporate beliefs about predispositions, including its own, in the same form as beliefs about other actions. Uncertainty about any predisposition on the part of a sentient agent represents a form of incomplete information — a form which occurs most often in the study of zero sum repeated games (see for example Bergin, 1992).

Hence the agent will have certain beliefs about the relative advantages to it of various outcomes, and for any individual *actor* these may differ between its agents. Thus the use of predisposition nodes in our theory allows us to represent the learning process an actor goes through as a result of his experiences in the game.

In the next section, we see how an agent's beliefs about its utility (incorporating information about its predisposition) relate to its *preference* over the possible outcomes of the game. Hence we have built in a mechanism by which the preferences of an *actor* can change over the course of a game. I believe that such a system is much more realistic than one in which the preferences of the actors are fixed at the start of the game, and never vary.

We have not addressed the question of the infinite hierarchy of beliefs, as described in chapter 2. Such an infinite hierarchy could be constructed by incorporating into an agent's prospective function its beliefs about another's prospective function, which would in turn include beliefs about that of the first agent, and so on. Now, there are ways of dealing with such an infinite hierarchy (see for example Mertens and Zamir 1985). But all such methods impose consistency conditions, such as coherence and a common prior, on the nature of actors' beliefs which I think are unduly restrictive. One of the most interesting questions in game theory is what happens when the beliefs of actors are inconsistent; by excluding that possibility game theory moves one step further from reality.

My approach has been quite deliberate: to avoid as far as possible the infinite hierarchy, and to ignore (at least for now) whatever cannot be avoided. This was done firstly by requiring the orientation graph to be common knowledge. Now of course common knowledge does itself represent an infinite belief hierarchy; what characterises common knowledge is that the infinite hierarchy is one of *certainty* rather than *uncertainty*. Thus it avoids the problems generally associated with the infinite hierarchy of beliefs.

Secondly, by restricting the domain of the prospective function to beliefs about objects in the set Q, we ignore the infinite hierarchy of beliefs about beliefs. This is not a deficiency in the model, and the reason for this is very simple. All an agent cares about are those aspects of the game which affect its utility, namely a subset of Q. And the only things which affect the objects in that subset and which the agent may experience are other objects in Q. Hence considering Q alone is sufficient to produce a complete theory.

Of course, in determining its prospective function, an agent may well take into account some hypothesis about what the other agents believe, and in that sense the infinite hierarchy still exists. But the only reason for an agent doing this is to assist in the refinement of its prospective function, which summarises the useful aspects of an agent's beliefs, namely its *marginal* beliefs about Q.

4.4 Belief Influence Diagrams

This chapter has seen the devlopment of two concepts: the construction of a graphical model in the form of the orientation graph and the formulation of a belief framework as embodied in the prospective function. We are now in a position to combine these two ideas to form a unified modelling framework for games. One further ingredient is needed to bind it all together, namely conditional independence.

4.4.1 Belief conditional independence

For this purpose we require a new version, known as belief conditional independence (b.c.i.).

Definition 4.15 *Let $X, Y, Z \subseteq Q$. Say: "Given Z, X is* belief conditionally independent *of Y" (notation $X \perp\!\!\!\perp_B Y|Z$) if it is common knowledge among all sentient agents that for all $\mathcal{X} \subseteq \mathcal{A}_X$, the prospective function $B_i(\mathcal{X}|(Y,Z) = (y,z))$ of any sentient agent π_i can be written as a function of Z alone.*

Formally, we define:

$$X \perp\!\!\!\perp_{B_i} Y|Z \quad \text{if} \quad B_i(\mathcal{X}|(Y,Z)) \perp\!\!\!\perp_V Y|Z[\mathcal{A}] \quad \text{for all } \mathcal{X} \subseteq \mathcal{A}_X, \text{ and}$$

$$X \perp\!\!\!\perp_B Y|Z \ , \quad \text{if for all } \pi_i \in \Pi_S, \text{ `} X \perp\!\!\!\perp_{B_i} Y|Z \text{'} \in K(\Pi_S).$$

We require that b.c.i. conforms to the same axioms C1–C3 as probabilistic c.i.

4.4.2 Definition of a BID

Definition 4.16 *A* belief influence diagram *(BID) I on a set of objects Q is a directed acyclic graph drawn in such a way that the following statements hold:-*

$$Q_i \perp\!\!\!\perp_B Q \setminus (Q_i \cup \mathrm{De}(Q_i)) | \mathrm{Pa}(Q_i) \ , \quad \text{for all } Q_i \in Q. \tag{4.1}$$

Thus a BID represents some subset of those c.i. relations between actions for which it is common knowledge that every sentient agent (according to its prospective function) *believes* them to be the case.

We now make the following assertion:

The orientation graph H is a belief influence diagram.

To start with, we note that it is sufficient to show that condition 4.1 holds for each of the four types of object in Q: actions by sentient agents, actions by agents of nature, predispositions and utilities. In addition, we recall that the orientation graph H is common knowledge to all sentient agents in the game. So any relations between objects in the graph must be relations of belief, commonly held by all agents. Hence any derived c.i. relations must be b.c.i. relations.

Now, consider the action A_j taken by a sentient agent π_j. From definition 4.4, $A_i \in \mathrm{Pa}(A_j)$ in $G = H|_G$ if and only if π_j admits A_i. So the set $\mathrm{Pa}(A_j)$ is exactly the set of actions known by π_j when it takes action A_j. An act a_j can only depend on what the agent knows when it takes action A_j, and what it believes about the things it does not know. In other words, given experience $\mathrm{Pa}(A_j)$, A_j is conditionally independent of every other act or predisposition which is not taken after it (in game time), namely the set $A \cup V \setminus (A_j \cup \mathrm{De}(A_j))$.

Similarly, we may argue that A_j is conditionally independent of every utility, the value of which is not realised after it, that is $U \setminus (A_j \cup \mathrm{De}(A_j))$, given that experience. Combining these two results gives us condition 4.1.

Next, consider an action A_j taken by an agent $\pi_j \in \Pi_N$. From Definitions 4.3 and 4.4, $A_i \in \mathrm{Pa}(A_j)$ in G if and only if the value $A_i = a_i$ is an input to the process which results in action A_j being taken by π_j: in other words, if and only if A_j is a (random) function of A_i. So, as above, A_j is conditionally independent of its non-descendants, given its parents, and condition 4.1 is satisfied. The same applies to a predisposition V_i, which is considered as an action taken by an agent of nature.

In the case of a utility U_i, from definition 4.9 we have

(iii) the utility nodes U are terminal nodes, that is $\mathrm{De}(U) = \emptyset$.

(iv) for all $a, a' \in \mathcal{A}_{A \cup W}$,

$$a|_{\mathrm{Pa}(U_i)} = a'|_{\mathrm{Pa}(U_i)} \implies U_i(a) = U_i(a').$$

Together these properties are equivalent to condition 4.1.

Thus our assertion is justified. There remains one further question; the non-uniquness of the BID. As with other forms of graphical model, not every set of b.c.i. relations can be represented in a single BID, and in general there is no unique minimal BID.

However, in an orientated game, the orientation graph is uniquely defined. Therefore, all we need do is specify that the BID of a game is that orientation graph, and is common knowledge.

Definition 4.17 *An orientated game Γ_A is said to be* belief-stuctured *if its BID I is the same as its orientation graph H, and I is common knowledge to all sentient agents.*

Henceforth, we assume that all games are belief-structured.

4.4.3 Assumptions about the model

In all its essential aspects, our model is now complete. We have defined a comprehensive framework for the graphical representation of the qualitative structure of a game. We now consider its implications.

While my aim is to keep the scope of the model as wide as possible, a number of important assumptions are unavoidable. To begin with, we restrict ourselves to the consideration of finite games only. As was explained earlier, this is not necessarily because the BID cannot be used to represent infinite games, but because the analysis of such games is beyond the scope of this thesis.

Next we have to fix the (partial) order in which actions are taken, and specify which agent takes them. As I argued earlier, these considerations do not for the most part prevent us from modelling games with a looser structure, but the graph of such a game may turn out to be overcomplicated. We must also specify the space of acts $\mathcal{A}_i$ available to every agent π_i. Although this is very important in mathematical terms, the practical implications do not pose us too much of a problem; if we wish to model a game in which an action space may vary (for example, depending on the value of another action), this can be taken care of simply by including additional actions.

Then we assume that there exist predispositions and corresponding utilities for each sentient agent, and that these are identical for every agent belonging to a given actor. On the first point generality is preserved, since we do not require that an agent knows its predisposition; indeed, we are able to model a situation in which an agent neither knows nor believes anything about its predisposition. There remains the question of whether a predisposition must necessarily exist. Such a philosophical question is beyond the scope of this thesis.

On the second point, there is an apparent loss of generality; I say apparent because I cannot imagine a situation in which we might want to model an actor as having agents with differing predispositions or utilities. We are, however, able to represent situations in which an actor has differing perceptions (or beliefs) of his utility over the duration of a game, simply by prescribing different prospective functions for his agents. The most important implication is that a sentient actor is identified with a particular set of agents, thus providing him with an identity.

Now let us consider the representation of belief in our model. With the exception of a few common knowledge conditions, an agent's beliefs are encompassed by its prospective function, which is restricted to beliefs about actions. As was explained earlier, this does not mean that agents do not have beliefs about other things, but merely that for the time being we do not need to include them in our model. Furthermore, the prospective function is capable of modelling just about any form of belief, whether vague or precise, ordinal or cardinal.

We defined intelligence with respect to an agent as meaning that it believes in what it knows to be true (that is it assigns a belief of 1), and does not believe in what it knows to be false (belief 0). Suppose an agent's prospective function contradicts this condition. Then it is effectively ignoring some piece of information or experience available to it. In other words, it does not admit (in either sense of the word) that action, and we should incorporate the consequent reduction of input in our model by leaving out the appropriate directed arc. Hence, without loss of generality, we may assume that every sentient agent is intelligent.

The most significant feature, however, of our treatment of belief is the nature of the prospective function itself. Unlike the other attributes of an agent, such as predisposition or utility, the prospective function is not represented explicitly in the BID. The reason is that the prospective function represents beliefs about all the objects in the BID. Inclusion of those beliefs as nodes in the BID would lead in general to an infinite hierarchy of beliefs: an unnecessary complication in my opinion. It would also contravene the principle of separation between belief and utility, as advocated for example by Savage (1954).

Clearly the trickiest part of the model is the identification of the orientation graph, based primarily on a partial ordering of actions, with the BID, based on a set of b.c.i. relations. Now, it is already specified that the orientation graph is common knowledge. Thus it is common knowledge that every sentient agent knows the structure of the graph, and hence, by the arguments set out above, it must be common knowledge that this is what every sentient agent believes.

But why should such beliefs necessarily take the form of a set of c.i. relations? Apart from the reasoning used above to justify this assertion, we can appeal to the well-documented link between graphical models and c.i., as described in chapter 3. It seems natural that this link should apply to any model of the same form, in terms of both the form of relation and the axioms to which it conforms. Indeed, Smith (1988) postulates that,

> "...any sensible concept of informedness between variables should exhibit such a c.i. stucture."

As far as I am concerned, all the other assumptions pale into insignificance beside the common knowledge conditions we impose. They represent a major restiction on the applicability of our model to any real-life situation. However, as I have said before, they are both minimal (less than in any other modelling system I know of) and necessary; the theory in the following chapters won't work without them.

4.4.4 Some comparisons

How does our construction of an influence diagram compare with that of others. Shachter (1986) introduced a distinction between arcs, with those directed into a decision node implying knowledge of a random variable, and all other arcs having the same pure c.i. interpretation as in a DAG. Smith (1989a) defined the same construction with greater rigour, incoporating a 'third party' decision analyst, for whom the client's decisions would also be random variables, thus facilitating a uniform interpretation of c.i. relations across all objects.

We have taken a fundamentally different approach. By considering random variables to be actions taken by agents (of nature), we have been able to utilise a uniform concept of 'influence', namely that of *admission*, to construct a partial ordering of actions. This partial ordering, as defined by the scheduling graph, forms the foundations of our model, and only later have we introduced a version of c.i.

Now, we could have taken a short cut and simply defined the BID in accordance with some set of b.c.i. relations. But by going the long way round, we have accomplished two things. The first is to justify the connection between the graph and b.c.i. The second, and most important, is to explain why agents in a game ought to have these sorts of beliefs in the first place.

For the essential distinction between our version of an influence diagram and that of Shachter (1986), and to a lesser extent Smith (1989a), is that ours represents the *perceptions* of the agents. This may or may not correspond to the *true* situation, providing of course that such a true situation exists. To summarise: while our definition of the BID corresponds with the usual ID formulation in so far as the b.c.i. relations follow an identical pattern, the interpretation of the

BID is very different.

Finally, I would like to compare the BID with the more usual graphical representation of a game: the extensive form. We have already done the equivalent comparison between the ID and the decision tree, and it is not unreasonable to suppose that the same advantages will obtain:

less complexity in the graph for models with symmetry;

the ability to model faithfully problems involving continuous variables;

computational efficiency;

the ability to accurately depict the qualitative structure of the model.

Now, at the end of chapter 2, I identified the inability to model complex problems as a deficiency in both the extensive and normal forms. The first three qualities listed above suggest that the BID represents a considerable improvement in this respect; examples to justify this claim are given later in the thesis. But its real strength as a modelling framework lies in its representation of qualitative structure.

It turns out that we can analyse a game using only the BID and a few simple principles. Such analysis allows us to simplify the game before any precise numerical distributions are introduced. Any simplifications we can make at the start represent a considerable reduction of effort, both in terms of the elicitation of beliefs and computation.

In chapter 5, I put forward a theory of sufficiency, based on the rationality of sentient agents; by introducing a principle of parsimony, we can deduce conditions for the BID under which an agent will rationally ignore certain information. In chapter 6, we consider a number of ways in which this theory can be used to simplify a BID.

5 Another Look at Rationality

So far, we have not said much about rationality, apart from the observation that there is no universally accepted definition of it. In this chapter we consider what 'rational' agents have in common that makes them rational. We attempt to construct a minimalist definition of rationality: some set of properties which we hope everyone can agree that a rational agent ought to display.

I see rationality as being fundamentally related to the concept of optimality; what is best for the agent ought to be rational for it as well. Before we can develop a theory of rationality, therefore, we must consider what is meant by optimality. We need to take into account the frame of reference: what makes an optimal action optimal. Then there is the question of how an agent translates its beliefs into an optimal action. In particular, can we find a set of rules which is sufficient to ensure that any action based on those rules will be optimal?

5.1 A Principle of Sufficiency

In this section, we develop a concept of sufficiency specifically related to the BID representation, and prove some technical results, showing under what conditions this form of sufficiency applies. When combined with some simple principles of rationality, these results provide a powerful tool for simplifying the structure of a game, as will be demonstrated in the next chapter.

5.1.1 Indifference

We wish to model how the beliefs and experience of an intelligent agent π_i affect the way in which it acts in a game. To start with, we will find it convenient to relax the prospective function notation.

From now on, we will use the term $B_i(X|Y)$ to denote the beliefs held by agent π_i about the *set of actions* X, given experience Y. The range of B_i is now unspecified, so $B_i(X|\cdot)$ could represent, for example, the set $\{B_i(\mathcal{X}|\cdot) : \mathcal{X} \in \mathcal{A}_X\}$, a single number $x \in \Re$ denoting the expectation of X, or simply a belief that $X = x$.

As we have already seen, when it takes action A_i the agent π_i holds only the following beliefs:

$$B_{A_i} := B_i(Q|\,\mathrm{Pa}(A_i))$$

that is its prospective function over the objects in the game given what it knows when taking an action A_i.

More specifically, we need to consider how the beliefs an intelligent agent has about its utility U_i affect its actions. Recall the definition of the orientation graph (definition 4.9), part (iv):

for all $a, a' \in \mathcal{A}_{A \cup W}$,

$$a|_{\mathrm{Pa}(U_i)} = a'|_{\mathrm{Pa}(U_i)} \implies U_i(a) = U_i(a').$$

This is equivalent to:

for all $a, a' \in \mathcal{A}_{A \cup W}$,

$$a|_{\mathrm{Pa}(U_i)} = a'|_{\mathrm{Pa}(U_i)} \implies B_i(U_i|a) = B_i(U_i|a').$$

In other words, indifference between two outcomes can be expressed in terms of equality of the prospective functions (over the agent's utility), having conditioned on each outcome.

Now, the above deduction involved a small slight of hand. Not only do we require that the agent is intelligent (that it believes its experience), but also that it believes a logical consequence of its experience. The two are not the same; for a discussion of this philosophical point in a game-theoretic context, see Binmore (1987). Nevertheless, we do not intend to invoke knowledge of *all* logical consequences as a general principle; we demand only that an intelligent agent has beliefs which are sufficiently coherent to guarantee the above statement applies.

We can naturally extend this idea to include equality of the prospective function with respect to utility, conditioned on the experience of any subset of actions in the game.

Definition 5.1 *For any subset of actions $X \subseteq A \cup W$, $x, x' \in \mathcal{A}_X$, an agent π_i is* indifferent between experience *x and x' ($x \sim x'$) if*

$$B_i(U_i|x) = B_i(U_i|x').$$

For example, if the agent's prospective function is represented by a zero-one possibility function, and if it treats uncertainty about utility within the context of a *Rawlsian* (minimax) framework, then it will be indifferent between x and x' if

$$\inf_{U_i|x} U_i = \inf_{U_i|x'} U_i,$$

where $\mathcal{U}_i|\cdot = \{u_i \in \mathcal{U}_i : B_i(u_i|\cdot) = 1\}$, and $\mathcal{U}_i$ is the *utility space* of agent π_i.

If the same treatment of utility is used, and the prospective function is in the form of a Dempster-Shafer belief function, then the condition for indifference is obtained by setting $\mathcal{U}_i|\cdot = \{u_i \in \mathcal{U}_i : \mathrm{Bel}_i(\mathcal{U}_i \setminus u_i|\cdot) < 1\}$ in the equation above.

For the case where the prospective function is expressed in terms of upper and lower previsions (denoted P_U and P_L), and π_i is *utilitarian* in outlook, the following conditions are necessary but not sufficient for indifference between x and x':

$$P_U(U_i|x) = P_U(U_i|x')$$

$$P_L(U_i|x) = P_L(U_i|x')$$

To guarantee indifference, we require that

$$P_U(U_i|x) = P_L(U_i|x) = P_L(U_i|x') = P_U(U_i|x'),$$

which is equivalent to the condition for indifference in the Bayesian framework, namely,

$$\mathbf{E}[U_i|x] = \mathbf{E}[U_i|x'],$$

where, in all cases, the expectations are taken over the agent's posterior beliefs about all the unknown objects $(A \cup W) \setminus X$ given x or x'.

Note that indifference (between outcomes) as expressed in terms of the equality of utilities may be considered a special case of indifference between experience. Henceforth, we shall refer to indifference between experience simply as 'indifference'.

We can now prove the indifference lemma:

Lemma 5.1 (Indifference Lemma) *Let A_i be the action taken by a sentient agent π_i. Suppose*

$$U_i \perp\!\!\!\perp_B R(A_i)|A_i \cup T(A_i)\ , \text{ where } R(A_i) \text{ and } T(A_i) \text{ partition } \mathrm{Pa}(A_i). \tag{5.1}$$

Then given any acts $r, r' \in \mathcal{A}_R$, $t \in \mathcal{A}_T$, and $a_i \in \mathcal{A}_i$, π_i is indifferent between (r, t, a_i) and (r', t, a_i).

Proof:

$$U_i \perp\!\!\!\perp_B R(A_i) | A_i \cup T(A_i)$$

$$\Longrightarrow \quad B_i(U_i | r, t, a_i) = B_i(U_i | r', t, a_i)$$

$$\Longrightarrow \quad \pi_i \text{ is indifferent between } (r, t, a_i) \text{ and } (r', t, a_i).$$

QED.

5.1.2 Preference

In general, an agent π_i may not know its own predisposition V_i. It will usually have some imperfect observation, or estimate, on which to base its choice of actions. We call this the agent's *preference*. Such a notion of preference is implicit in the *axiom of revealed preference*:

> An agent's preference can in principle be completely determined by observing its actions in every possible situation.

However the axiom of revealed preference is not uncontroversial, and we do not demand it as part of our theory.

Our definition is not based on some fixed preference ordering determined at the start of the game, to which role we have already assigned the predisposition function. The preferences of an agent are determined according to its experience, including any knowledge it may have about its own predisposition function. Thus the agents belonging to an individual actor may have different preferences. This concept of preference allows us to model the changing of an *actor's* preferences over the duration of a game as he learns about his predisposition function.

First of all, consider the special case where the experience subset of an intelligent agent π_i is $X = A \cup W$, and $V_i \in W$. Then we can define the natural preference relation over all such subsets $x, x' \in \mathcal{A}_X$:

> x is preferred by π_i to x' if and only if $U_i(x) > U_i(x')$.

We now extend this preference relation to proper experience subsets in the same way as this was done for indifference. For this we require at least a partial ordering of the marginal prospective function $B_i(U_i|X)$, $X \subseteq A \cup W$, of an agent π_i. This, when combined with indifference between experience, according to the above definition, induces a partial preference ordering over experience.

Definition 5.2 *We say a sentient agent π_i prefers experience x to experience x' ($x \succ x'$) if*

$$B_i(U_i|x) > B_i(U_i|x').$$

A complete ordering, such as that already defined over the set $X = A \cup V_i$ will not in general exist for proper experience subsets, and the partial preference ordering may be very limited in domain. In the case where beliefs are expressed in terms of a zero-one possibility function, the induced partial preference ordering will in general be null. There are a few exceptions such as when the agent uses a minimax decision rule, in respect of which it prefers x to x' if,

$$\inf_{\mathcal{U}_i|x} \mathcal{U}_i > \inf_{\mathcal{U}_i|x'} \mathcal{U}_i.$$

If upper and lower previsions are being used then the agent prefers x to x' if

$$P_L(U_i|x) \geq P_U(U_i|x'), \text{ and}$$

$$P_U(U_i|x) > P_L(U_i|x').$$

This will in general produce a non-trivial partial ordering.

On the other hand, if π_i is a Bayesian, a complete preference ordering can be defined according to expectation over the relevant conditional distribution:

$$x \succ x' \iff \mathbf{E}[U_i|x] > \mathbf{E}[U_i|x'],$$

for all x, x' which are assigned positive prior probability.

Preference of experience is quite a unusual concept in the context of game theory. So let us pause for a while to consider what this partial preference ordering over experience really means. We can relate to it by considering phrases like, 'I would have preferred it had so-and-so done something else,'. It is as if we were to stop the game part way through, and considered how well it was going for some agent. A similar concept, the 'value of wizardry', is discussed by Matheson (1990).

For the purposes of the theory which follows, we only need to compare two experiences in the same space $\mathcal{A}_X$. But there is in principle no reason why we could not have,

$$B_i(U_i|x) > B_i(U_i|y), \text{ with } \mathcal{A}_X \neq \mathcal{A}_Y.$$

As with indifference, preference of one outcome over another, as defined by the first having a greater utility, may be considered a special case of preference of experience. From now on, we shall refer to preference of experience simply as 'preference'.

5.1.3 Optimality

But how do such preferences affect the actions taken? Well, suppose π_i has experience x immediately before taking action A_i. And suppose that for some $a_i, a_i' \in \mathcal{A}_i$, π_i prefers $x \cup a_i$ to $x \cup a_i'$. Then given x, we can deduce that it ought to prefer act a_i to a_i'. Thus, a partial preference ordering over experience induces a similar ordering over acts.

Definition 5.3 *Suppose agent π_i has experience x when taking an action A_i. Then define a partial preference ordering over $\mathcal{A}_i$ as follows:*

For all $a_i, a_i' \in \mathcal{A}_i$,

$$a_i \succ a_i' \iff (x \cup a_i) \succ (x \cup a_i').$$

$$a_i \sim a_i' \iff (x \cup a_i) \sim (x \cup a_i').$$

Definition 5.4 *Given experience x, define an act $a_i^* \in \mathcal{A}_i$ to be* optimal *if for all $a_i' \in \mathcal{A}_i$,*

$$a_i^* \succsim a_i'.$$

Note that, as with experience, there will only exist in general a partial preference ordering over $\mathcal{A}_i$, and hence an optimal act may not exist. Furthermore, even if an optimal act does exist, it may not be unique.

We can now state and prove the main technical result of this section.

Theorem 5.2 (Optimality Theorem) *Let A_i be the action taken by a sentient agent π_i. Suppose there exist sets $R(A_i)$ and $T(A_i)$ satisfying condition 5.1, as given in lemma 5.1.*

Let $r, r' \in \mathcal{A}_R$, and $t \in \mathcal{A}_T$, and suppose that given experience (r, t), there exists an optimal act $a_i^* \in \mathcal{A}_i$. Then a_i^* is also optimal given experience (r', t).

Proof: Consider any act $a_i' \in \mathcal{A}_i$. By Lemma 5.1, π_i is indifferent between (r, t, a_i^*) and (r', t, a_i^*). Similarly, π_i is indifferent between (r, t, a_i') and (r', t, a_i').

Thus we have:

$$(r', t, a_i^*) \sim (r, t, a_i^*) \succsim (r, t, a_i') \sim (r', t, a_i').$$

Hence a_i^* is an optimal act given experience (r', t).

QED.

So we have found a rule (condition 5.1) which tells us under what cicumstances, given some subset of experience T, the set of optimal acts is invariant with respect to ~~a~~ the remainder R. Moreover, since this rule is defined entirely according to the BID, we do not need to know the precise prospective function; in particular, the relationships between variables in the game including the agent's utility are irrelevant.

As far as the agent π_i is concerned, this is sufficient. Since it will always know $T = t$ when it takes action A_i, it will be able to deduce that the set of acts available to it which are optimal is invariant with respect to R.

However, this is not in general true for the other agents. An agent whose experience does not include T cannot make the same deduction. For this, we need to consider what rule an agent is using to decide its act, and whether that rule could be described as optimal.

5.1.4 Policies

Every agent π_i, whether or not it is rational, intelligent or even sentient has to have some process which determines which act it chooses given any particular admission, and any beliefs in the case of a sentient agent. Now, we will not attempt to define every possible such process; the class would need to include any process which might be employed by, say, a 'crazy' sentient agent.

Instead, we consider the simplest form of process, namely a function which, given any admission, determines a single act and causes that act to be chosen. Such functions are elementary to every action process an agent might use. We call this function a policy.

Definition 5.5 *A* policy $S_i = S_i(x) \in \mathcal{S}_i$ *for the action A_i taken by an agent π_i is a function which induces an act $a_i \in \mathcal{A}_i$ given any admission $X = x$, $X \subseteq A$. The* policy space $\mathcal{S}_i(X) = \mathcal{A}_X \otimes \mathcal{A}_i$.

How may policies be used by an action process? For an action taken by an agent of nature, in other words a variable, there are two possibilities. If the variable is determinate, then its distribution is defined by a single policy. If it is stochastic, then the process will involve some form of randomisation over several policies. A sentient agent can choose to adopt either of these processes, although other options are possible.

Recall that by definition, when taking action A_i, π_i is restricted to experience $\text{Pa}(A_i)$, so we can simplify this definition of a policy by specifying a_i only for the cases where $X \subseteq \mathrm{Pa}(A_i)$. For the rest of this section we assume $x \in \mathcal{A}_{\mathrm{Pa}(A_i)}$.

Note the distinction made here between a policy S_i, which specifies the act $A_1 = a_i$ to be chosen by an individual agent π_i and a *strategy*, which is the term used by most game-theorists to describe the set of policies $\{S_i : \pi_i \in \Pi_j\}$ employed by a sentient actor Π_j for each action he takes in the game.

5.1.5 Optimal policies

From the arguments and definitions above, it is easy to see how to define a partial preference ordering over the set of policies $\mathcal{S}_i$.

Definition 5.6 *We say that agent π_i is* indifferent between policies *S_i and S_i' ($S_i \sim S_i'$) if*

$$S_i(x) \sim S_i'(x)$$

for every $x \in \mathcal{A}_X$.

We say that π_i prefers *policy S_i to policy S_i' ($S_i \succ S_i'$) if*

$$S_i(x) \succsim S_i'(x)$$

for every $x \in \mathcal{A}_X$, and there exists an $x \in \mathcal{A}_X$ such that,

$$S_i(x) \succ S_i'(x).$$

We say a policy S_i^* is optimal *if for all $S_i' \in \mathcal{S}_i$,*

$$S_i^* \succsim S_i'.$$

So, for example, a minimax optimal policy S_i^*, with the prospective function defined in terms of a zero-one possibility function has to satisfy:

$$\inf_{\mathcal{U}_i|(x, S_i^*(x))} \mathcal{U}_i \geq \inf_{\mathcal{U}_i|(x, S_i'(x))} \mathcal{U}_i,$$

for all $S_i' \in \mathcal{S}_i$ and all x. A similar result is obtained for minimax with a Dempster-Shafer belief function.

S_i^* is optimal within the utilitarian framework if,

$$P_L\left(U_i|x, S_i^*(x)\right) \geq P_U\left(U_i|x, S_i'(x)\right),$$

where the prospective function is expressed in terms of upper and lower previsions, and within a full Bayesian model if,

$$\mathrm{E}\left[U_i|x, S_i^*(x)\right] \geq \mathrm{E}\left[U_i|x, S_i'(x)\right],$$

again for all S_i' and x, where all expectations are defined in the usual way.

Corollary 5.3 (Sufficiency Principle) *Let A_i be the action taken by a sentient agent π_i, and let there exist sets $R(A_i)$ and $T(A_i)$ satisfying condition 5.1.*

Suppose there exists an optimal policy $S_i^* \in \mathcal{S}_i$. Then there is an optimal policy $S_i' \in \mathcal{S}_i$ such that,

$$S_i'(r,t) = S_i'(r',t)$$

for all $r, r' \in \mathcal{A}_R, \quad t \in \mathcal{A}_T$.

Proof: Choose any $r \in \mathcal{A}_R$, and define the policy S_i' as follows:

$$S_i'(r',t) = S_i^*(r,t) \text{ for all } r' \in \mathcal{A}_R,\ t \in \mathcal{A}_T.$$

Now, from definition 5.6, the act a_i^* induced by $S_i^*(r,t)$ is optimal for all $r \in \mathcal{A}_R$, $t \in \mathcal{A}_T$. And by theorem 5.2, a_i^* is also optimal given experience (r',t). In other words, as the act induced by $S_i'(r',t)$, a_i^* is optimal. Hence S_i' is optimal.

QED.

We now have the condition we need. Since the BID is common knowledge to all sentient agents, they will all be in a position to deduce that the set of optimal policies available to the agent π_i will be invariant with respect to R *whatever* T it experiences. What makes the sufficiency principle so powerful is that it depends only on the BID. Thus, as we will see in the next chapter, it can be used to make important general deductions about a game before any precise numerical relationships have even been specified.

There are two things to note about the above corollary. Firstly $T(A_i)$ is known as a sufficient set for $\mathrm{Pa}(A_i)$ (with respect to π_i taking action A_i). It is not unique, and in general even a unique minimal sufficient set may not exist. Secondly the sufficiency principle as given above relates only to optimal policies. Although optimal policies may be considered desirable in general terms, we cannot rule out the possibility that a sentient agent may not adopt such a policy.

In the next chapter, we will impose uniqueness of the sufficient set as a condition for the BID being a 'good' model of the game. We will show also that the Sufficiency Principle does not rely on optimality, and can apply under far more general conditions.

5.2 Rationality

In the previous section, we said that a rational agent π_i ought to have some partial preference ordering over the set of acts available to it given its experience. In other words, for a given action A_i, it should have a partial preference ordering over the set of policies $\mathcal{S}_i$.

In this section we will extend this idea of rationality. We will examine various forms of rationality, and the consequences of imposing rationality on one or more of the agents in a game. We divide rationality into two forms, which we denote rationality of belief, and rationality of action.

5.2.1 Rationality of belief

Rationality of belief involves *internal* rationality (coherence of belief and avoidance of sure loss) and *external* rationality (relating beliefs to the available evidence). For now we will restrict our consideration of rationality of belief to what we have already discussed in the previous chapters. With respect to external rationality, we will require that a rational agent be intelligent. Recall the conditions for intelligence:

An agent π_i is said to be *intelligent* if

B3 $B_i(x|x,y) = 1$ and

B4 $B_i(x|x',y) = 0$ if $x \neq x'$,

provided every term is well defined.

Now these conditions are not particularly strong, since they do not guarantee that the beliefs held in such situations reflect certainty about some event on the part of the agent. This in turn is due to the fact that the form of a prospective function as defined may not be capable of expressing such certainty. I suggest that any prospective function which is unable to express certainty is seriously deficient.

Therefore from now on we will assume that every prospective function B_i held by a sentient agent π_i is sufficiently flexible that it can accommodate an expression of certainty for any $\mathcal{X} \subseteq \mathcal{A}_X$, for every subset $X \subseteq Q$. I do not believe that this stipulation involves too great a loss in generality. In particular, every form of prospective function considered so far in this thesis (and I expect, every type of belief ever considered in a game-theoretic context) qualifies under this condition.

Given this requirement, adherence to conditions B3 and B4 will guarantee that experience implies certainty, since 0 and 1 represent ~~the~~ respectively the infimum and supremum of degree of belief for every prospective function, under the restricted definition which applies to those conditions. Note that we do not demand that every sentient agent will necessarily believe anything with certainty, merely that it is capable of so doing.

Intelligence relates specifically to external rationality. But what about internal rationality, or coherence of belief? So far we have demanded that the BID be common knowledge to all sentient agents. We now extend this to include, explicitly, common knowledge of every logical consequence of that BID, namely the set of c.i. statements which may be deduced from that BID using the axioms C1–C3, or equivalently the d-separation theorem. Thus any beliefs held by a sentient agent must be consistent with these c.i. statements.

These are the only coherence conditions which we demand. Any other form of coherence will manifest itself in the form of the particular type of prospective function an agent has; for example, a Bayesian prospective function demands very strict coherence conditions.

Finally, we consider the possibility that beliefs could be linked to utility in some rational way. The classic example of this is Pascal's wager, as discussed by Savage (1954) for instance, in which it is postulated that it may be rational to believe in God due to consideration of the possible outcomes given belief and non-belief. While there are many ways of countering this argument, I reject it simply on the grounds that an agent cannot choose its own beliefs. For an extended discussion on rationality of belief, see for instance Walley (1991).

5.2.2 Rationality of action

Given that our model is based primarily on the actions taken by agents, and that the prospective function defines beliefs about those actions, it seems that rationality of action will prove to be more fundamental to our argument than rationality of belief. Therefore we will treat the two topics differently. We will consider rationality of action in far greater detail. In addition, whereas we simply assume the conditions required for rationality of belief, the conditions for rationality of action will form the basis of our definition of rationality itself; in other words, our definition of rationality will be defined in terms of *behaviour*.

Informally, we will require that a *rational* agent π_i satisfies the following criteria:

(i) If a set of optimal policies $\{S_i^*\} \subseteq \mathcal{S}_i$ exists, it should employ one of them.

(ii) The partial preference ordering over $\mathcal{S}_i$ should be rich enough to accommodate such an optimal policy set.

As we have already seen in chapter 2, there is no agreed definition of rationality in the game-theoretic context. And the form of rationality chosen is very often related to the particular solution concept under investigation rather than any higher ideal.

I am of the opinion that there should be a fundamental definition of rationality applicable to every type of game. Any further behavioural attributes should not be defined according to some code of 'rationality'. Instead they will be determined by the personality of the agent, and the situation in which it finds itself, that is by its experience. Note that we do not implicitly restrict every agent to rational behaviour; rationality of action should be considered a norm for the agent in question, without asuming that any other agent will necessarily conform to it.

Our definition of rationality is *inclusive*, rather than *exclusive*. In other words, we aim to include only those attributes which are both applicable to all games and relevant to our approach. Therefore it should not be seen as a definitive list, any attempt at which I believe to be futile, but as a small part of a much wider picture.

So how do our two criteria fit the bill? Well, the first criterion seems to be the obvious starting point; in some form or other it is included either explicitly or implicitly in just about every version of rationality. I would argue (and it seems most would agree) that this criterion is necessary for any meaningful definition of rationality. Indeed, if we accept the axiom of revealed preference, which implies that when acting, an agent does whatever it thinks is best at the time, it may be considered a tautology.

The second is perhaps less obvious. It requires that the agent can compare every policy available to it with some 'best' policy in order to confirm that the policy in question is indeed optimal. And with policy spaces being infinite in many cases, why should we expect an agent to be capable of this? Nevertheless we will require this condition to hold, since otherwise how is an agent to choose between policies?

We do not specify, however, to what extent this partial ordering is determined scientifically (that is through calculation). A rational agent is still permitted to use judgement and guesswork in formulating optimal policy — indeed this would be the case in virtually all 'real-life' situations. In practice, if after reflection an agent is unable to rank two or more candidates for optimal policies, then they must be so nearly equally desirable that little could be lost by setting them as equivalent within the partial preference ordering.

Definition 5.7 *Call an agent π_i* rational *if it employs an optimal policy $S_i^* \in \mathcal{S}_i$.*

5.2.3 Choosing between optimal policies

However, we still have a problem. In general there may be many equally attractive policies to choose from. We may assume either that such policies are transparently equally preferable, or that no amount of deliberation on the part of the agent can separate them. Therefore according to the above definition, a rational agent must satisfy an additional criterion:

(iii) The agent π_i should have some way of choosing between several optimal policies.

This is not so much concerned with rationality as with being decisive; if an agent has no way of choosing between optimal policies, nothing happens — and that contradicts the definition of a game. Rationality has traditionally had a lot to say about making well-considered judgements, and not making over-hasty decisions. In contrast, this criterion can be summed up by the phrase:

> 'thinking is all very well, but eventually you must *do* something'.

However, this piece of advice does not help very much, unless we have some method by which a rational agent can choose between two or more equally good options. Various solutions have been proposed, all of which have problems. The choice could be arbitrary, since by definition it doesn't matter which is chosen, but that avoids the question instead of answering it.

The choice could be made by some random mechanism within the agent's thought processes, that is by using a *mixed strategy*, although it is not clear whether it is possible for the human brain to perform this sort of meta-rational mental convolution. Or there could be an explicit random process, such as tossing a coin, to determine which act to choose, but some (eg. Howard 1971) see this as 'irrational'. Moreover, if a random mechanism (either internal or external) is to be used, it is still left to determine what distribution to employ, and we are back to square one.

We do not pretend to be able to resolve this question completely. Nevertheless, we can offer a partial solution which at least goes some way towards reducing the available options, via a principle of parsimony.

5.3 Parsimony

In this section, we will consider the role of *information* in a game. Our use of the term information will refer both to the admission, knowledge and experience of agents in a game and to the related concept of how we model that game.

5.3.1 Relevant information and modelling assumptions

Consider the action A_i to be taken by the sentient agent π_i. In principle, any experience π_i has when taking A_i will be in $\mathrm{Pa}(A_i)$, even if this information has no bearing on the game. Of course, such irrelevant information should not be included in the model in the first place. But there are two questions which arise:

Firstly, who is to say what is relevant and what is not? So far we have laid down no conditions on this aspect of the game. Therefore as things stand, it is up to the individual agent to decide. However, we require a common interpretation of the game by all agents, as expressed in the BID, since what an agent chooses to take into account when taking an action will typically depend on what it believes the other agents are going to do.

And secondly, on analysing a game — for that is after all what this thesis is about — we may find that some action which was included in the original model, may turn out to be irrelevant after all. In other words, all agents in the game will act in exactly the same way (under our model) whether that action is included or not.

Of relevance to the first question is the concept of *common knowledge* which we have already discussed in chapter 2. To help answer the second, we now turn our attention to providing some inclusion criteria for the model, and for simplifying it as a result of analysis.

5.3.2 A principle of parsimony

We require a way of distinguishing between relevant and irrelevant information. In addition, we will require that such a distinction be expressed in terms of the behaviour pattern of a 'rational' agent. Such an expression of rationality is clearly going to relate most closely to criterion (iii) above:

> The agent π_i should have some way of choosing between several optimal policies.

From the sufficiency principle, we see that if an agent π_i knows that some action $A_j \in \mathrm{Pa}(A_i)$ can only ever affect its utility U_i via its own action A_i, then given an optimal policy q_i^* which depends partly on A_j, it can always find another optimal policy q_i' which does not depend on A_j. Therefore it would seem sensible in this situation for π_i effectively to ignore A_j when taking action A_i. It would then be conforming to a principle of parsimony.

Parsimony relates to the simplicity of a model, often measured in a statistical context by the number of parameters involved. It is implicit throughout the whole of statistical modelling theory (see for instance McCullagh and Nelder (1983) for an example of the use of parsimony in this context) and scientific modelling in general. Our definition of parsimony is firmly within that tradition, although we emphasise a more personal aspect of it. In particular, we will show how it can be applied not just by a statistician trying to model a situation, but to the thought processes and behaviour of an individual who is being modelled.

For example, following the Bayesian paradigm, an agent π_i, having marginalised out all its beliefs about future actions, might find that its payoff was (belief) conditionally independent of the action A_j ($\in \mathrm{Pa}(A_i)$) given A_i and $\mathrm{Pa}(A_i) \setminus A_j$. Regardless of the form of utility function being used by π_i, its optimal act would not differ, whatever the observed value a_j.

Definition 5.8 *Suppose the policy space for an action A_i depending on experience $\mathrm{Pa}(A_i)$ is $\mathcal{S}_i = \mathcal{A}_{\mathrm{Pa}(A_i)} \otimes \mathcal{A}_i$, and suppose there exist subsets $R(A_i)$ and $T(A_i)$ satisfying condition 5.1.*

Define the sufficient policy subspace *for* $T(A_i)$ *to be* $\mathcal{T}_i = \mathcal{A}_{T(A_i)} \otimes \mathcal{A}_i$, *with each policy* $S_i' \in \mathcal{T}_i$ *specifying an act* $a_i \in \mathcal{A}_i$ *for every possible experience* $t(A_i) \in \mathcal{A}_{T(A_i)}$.

Definition 5.9 *Call a policy* $S_i' \in \mathcal{T}_i$ at least as simple as *a policy* $S_i \in \mathcal{S}_i$ *(say* $S_i' \subseteq S_i$*) if* $\mathcal{T}_i \subseteq \mathcal{S}_i$.

Clearly, simplicity here may depend on the parameterisation of the game. For example, if

$$\begin{aligned} \mathrm{Pa}(A_i) &= (A_j, A_k), \text{ and} \\ \mathrm{Pa}(A_i') &= (\tfrac{1}{2}A_j + \tfrac{1}{2}A_k, \tfrac{1}{2}A_j - \tfrac{1}{2}A_k), \end{aligned}$$

then the 'simple' policies may differ. However, given any particular BID, simplicity is well defined; one policy is at least as simple as a second policy if the set of nodes representing the parameters of the first policy form a subset of the corresponding set of nodes for the second. This is an important advantage associated with the BID representation.

Definition 5.10 *Let* $\{S_i^*\} \subseteq \{\bigcup \mathcal{T}_i : \mathcal{T}_i \subseteq \mathcal{S}_i\}$ *be a set of policies for the action* A_i *between which the agent* π *is indifferent. Then* π_i *is* parsimonious *if it does not choose a policy* $S_i \in \{S_i^*\}$ *when there is a simpler policy* $S_i' \in \{S_i^*\}$.

Note that if $S_i \not\subseteq S_i'$ and $S_i' \not\subseteq S_i$ then both may be considered candidates by a parsimonious agent. In section 5 we shall impose conditions on the model which prohibit this from occurring, whereupon the above definition simplifies to:

> An agent π_i is parsimonious if, given a choice between policies in $\{S_i^*\}$, it chooses a policy which is at least as simple as every policy in $\{S_i^*\}$.

5.3.3 Parsimony in action

So why choose parsimony as a criterion for distinguishing between policies — isn't it just another arbitrary way of reducing the set of equally good policies? I believe this is not the case, and some of the reasons why have been alluded to above.

Firstly, parsimony represents a way of simplifying our model; remember that, although we are trying to model the thought processes of an agent via the policy it adopts, the other agents in a game will observe not that policy, but the consequent act.

Definition 5.11 *Suppose policy $S_i' \in \mathcal{T}_i$ is at least as simple as $S_i \in \mathcal{S}_i$. Say they are* equivalent *if for all $X \in \mathcal{A}_{\mathrm{Pa}(A_i)}$,*

$$S_i'(X \mid_{\mathcal{T}_i}) = S_i(X \mid_{\mathcal{S}_i}).$$

Thus if two policies are equivalent, another agent will be unable to tell which policy the agent π_i is employing, whatever act i obtains, since that act will always be the same under both policies. We now show that any policy which is equivalent to an optimal policy is itself optimal.

Proposition 5.4 *If there exists an optimal policy $S_i' \in \mathcal{T}_i$ which is at least as simple as an optimal policy $S_i \in \mathcal{S}_i$, then the equivalent policy to S_i in $\mathcal{T}_i$ is also optimal.*

Proof: Consider an optimal policy $S_i \in \mathcal{S}_i$, where $\mathcal{S}_i = \mathcal{A}_i \otimes \mathcal{A}_X$ for some $X \in \mathrm{Pa}(A_i)$, and $S_i(x) = a_i$. Suppose there is another optimal policy $S_i' \in \mathcal{T}_i = \mathcal{A}_i \otimes \mathcal{A}_Y$ for some $Y \subseteq X$. Then setting $y = x \mid_{\mathcal{A}_Y}$, the policy S_i'' defined by:

$$\begin{cases} S_i''(Y) = a_i & \text{if } Y = y, \\ S_i''(Y) = S_i'(Y) & \text{if } Y \neq y, \end{cases}$$

is also optimal.

It follows that the policy $S_i''' \in \mathcal{T}_i$ defined by:

$$S_i'''(X \mid_{\mathcal{A}_Y}) = S_i(X), \quad \text{for all } X \in \mathcal{A}_X$$

is optimal.

QED.

Proposition 5.5 *Conversely, for any optimal policy $T_i \in \mathcal{T}_i$, there is an equivalent optimal policy $T_i' \in \mathcal{S}_i$.*

Proof: Set $T_i'(X) = T_i(X \mid_{\mathcal{T}_i})$, and the result follows from the proof to proposition 5.4.

QED.

Hence, an agent will observe the same act made by another agent, whichever equivalent optimal policy it uses. So in general, it is impossible to tell from an agent's actions whether or not it is parsimonious, and in fact it makes no difference to the outcome of the game.

Therefore, for the sake of model simplicity, we should eliminate any unnecessary dependencies, thus enforcing parsimony in the normative sense. This is done using a process known as arc deletion, which we will describe in chapter 6. The strength of this result arises from the fact that the prospective function of an agent incorporates all relevant information within the agent's beliefs about the actions in a game.

Secondly, we might want to consider the 'cost' incurred by agents in remembering or thinking about irrelevant information. Now you may say that such costs should already have been included in the model. However in order to model this within the B.I.D. we need to introduce additional nodes representing the decisions on whether or not to consider additional pieces of information, and we are back to square one. And even within alternative modelling systems, the problem of infinite regression is encountered (that is ... the cost of thinking about the cost of thinking about the cost of ...). When costs or other 'external' restraints are explicitly included in the model, we enter the realm of bounded rationality, which was briefly touched upon in section 2.3.2.

In similar fashion, parsimony is consistent with the aim of choosing a policy which takes less time to calculate; under the scheme for calculating optimal policies proposed by Binmore (1988), this would tend to result in a better policy being chosen.

5.3.4 Parsimony in practice

But there is a more fundamental argument as to why parsimony should play a part in the way agents behave. According to the Bayesian school (see Savage, 1954), the act to be taken should depend only on the expected utility of that act, given the currently available information. Taken at face value that excludes the concept of parsimony, since we are introducing an additional factor into the decision-making process.

But parsimony is *implicitly* used in the formulation of a model, and this is true for every other proposed system of decision theory. Indeed, a stronger version of the parsimony principle (see chapter 6) underpins the scientific method itself, since without some limitation of the scope of a problem, model formulation is impossible. By including parsimony as an axiom, we are merely making *explicit* what is in fact common practice in the modelling of uncertain environments.

One objection to parsimony might be that it disallows the explicit use of 'mixed strategies', since such a strategy makes use of some random mechanism, which is known only to the agent concerned, and is otherwise inconsequential as regards the outcome of the game. Mixed strategies are ubiquitous in game theory and, although their use is disputed (see for example Howard, 1971), their existence still underpins many game-theoretic results.

We have already dealt with this question in chapter 2; to repeat, the importance of the mixed strategy in game theory relates not to the fact that anyone actually uses such a device (and if they did, it would be impossible to detect in any case), but to the *uncertainty* of other agents about which 'pure strategy' is being employed.

In Chapter 6, we will demonstrate a variety of ways in which the concepts of sufficiency, parsimony and various forms of rationality can be used to simplify our model of a game, as defined by the BID.

6 Simplifying the Model

The BID I of a game includes every factor known to an agent which could possibly influence its outcome. As as result it can become very large. In this chapter we will consider a variety of methods which may be used to simplify a BID and hence simplify the model which it represents.

The behaviour of an agent must depend not only on the extent of its own 'rationality' but on its assessment of the 'rationality' of all the other agents and, by extension, on its assessment of their assessments of the 'rationality' of everyone else, and so on ad infinitum. Hence common knowledge will play an important part in our analysis. In particular, we require common knowledge to simplify a game. Recall Definition 4.17:

> An orientated game Γ_A is said to be *belief-structured* if its BID I is the same as its orientation graph H, and I is common knowledge to all sentient agents.

Therefore any simplification of the game which involves altering its BID must also be common knowledge to all sentient agents, as must the reasoning upon which that simplification is based. Later on we will consider how a game may be simplified via a process of arc deletion. To begin with with, however, we will consider node reduction, which reduces the complexity of a model by eliminating unwanted actions or parameters.

6.1 Node Reduction

The BID may include nodes in which we are not really interested: in other words, nodes representing actions which are irrelevant to all agents, given common knowledge that all are parsimonious. How can we tell which nodes are irrelevant and which are not, and what assumptions do we have to make to justify removing them from the BID? To answer this question, we must first consider how the BID should be drawn initially to model a game both faithfully and usefully.

6.1.1 Suppressing predisposition nodes

The first issue connected with the modelling process which requires consideration is the treatment of predisposition, and in particular how it is represented in a BID. Recall that in defining the orientation graph, and hence the BID, we have chosen only a subset W of the set of predispositions V to be represented by nodes in the graph. We must now consider what criteria can be applied to decide W.

Now, suppose that some predisposition V_i is common knowledge to all sentient agents, represented in a BID by V_i being a parent of every action taken by a sentient agent. This is equivalent in the usual game theory terminology to some part of the (normal form) payoff matrix being common knowledge, an extremely common assumption in game theory. Then there is no modelling requirement to include V_i in the BID, since such common knowledge can be incorporated in every agent's prospective function without compromising the accuracy of the model.

Next, consider the opposite extreme: that no sentient agent (not even π_i itself) admits V_i, and that no sentient agent can infer anything about V_i from any action which it does admit. Such a situation might be represented by a BID in which for every sentient agent π_j, $A_j \perp\!\!\!\perp_B V_i$ unconditionally. In this case also we would want to omit the node V_i from the BID, since it could simply be incorporated in U_i.

In fact, we can generalise these cases; if every sentient agent falls into one of the two categories described above, then we should not include V_i in the BID. The general requirement if a predisposition node V_i is to be omitted is that no sentient agent has access to any information about V_i unless it admits V_i. In other words, no agent can learn about V_i, other than by observing it. We will consider the concept of learning in more detail in the next chapter.

Recall that we treat predisposition nodes as actions of nature when defining their relationships to other actions in the BID. So in fact we can generalise the above analysis to include all actions of nature.

How can we put these principles into practice? Since we are talking essentially about how we as statisticians or game theorists choose to model a game, there can be no hard and fast rules; it is essentially a question of good judgement (based on the scientific method). Nevertheless, suppose that during the process of modelling a game, a preliminary version of the BID contains an *orphan* node, that is a node with no parents, satisfying the above conditions. Then that node may be supressed by removing it and all connecting arcs from the BID. Successive application of this principle may be used to substantially reduce the size of a BID. However, such suppressions are to be used with extreme caution. Specifically, they must be justified in terms of the rationality of the agents in a game.

The simplifications of the BID described above are based more on practical than theoretical considerations. When the BID of a game is drawn, in practice we will not include absolutely everything which is known to the agents as a node in its own right. Instead such knowledge will be incorporated into the prospective function of each agent. As outlined in chapter 5, certain unknown objects may be deemed irrelevant to the game. Again these should not be included, and will not feature in any agent's prospective function, since they make no difference to the set of optimal policies.

6.1.2 Barren node reduction

The previous manipulations involving the removal of unwanted nodes are based on principles of statistical modelling and common sense. We now describe a further way of deleting nodes which is based on the BID as already specified.

Definition 6.1 *A* barren node A_i *is an action which has no descendents in I, or whose descendents are all barren. So if* Ba *is the set of barren nodes,*

$$\mathrm{Ba} = \{A_i \in A : \mathrm{De}(A_i) \subseteq \mathrm{Ba}\},$$

or equivalently,

$$\mathrm{Ba} = \{A_i \in A : A_i \notin \mathrm{An}(U)\}.$$

Under what conditions will such actions have no effect on any of the other actions in the game, or on the utility of any agent?

We will argue that, as a corollary of the axiom of revealed preference, a sentient agent π_i will disregard actions represented by barren nodes. Furthermore, provided it is common knowledge to all sentient agents that every such agent will behave in this way, then barren node actions become irrelevant to the game, and can be removed from our model.

Case 1: $A_i \notin \mathrm{Ba}$, π_i sentient.

Suppose that when π_i takes its action A_i, it takes into consideration the existence in the game of an action $A_j \in \mathrm{Ba}$ in such a way as to affect its own choice of act $a_i \in \mathcal{A}_i$. Then π_i is exhibiting a *revealed preference* over $\mathcal{A}_j$ for some combination $a_{-j} \in \mathcal{A}_{-j} = \bigotimes_{A_i \in A \setminus A_j} \mathcal{A}_i$. Thus it believes that for some $a_j' \in \mathcal{A}_j$,

$$U_i(a_{-j}, a_j) \neq U_i(a_{-j}, a_j').$$

But $A_j \in \text{Ba}$, contradicting the definition of the BID.

Case 2: $A_i \notin \text{Ba}$, $\pi_i \in \Pi_N$.

$\text{Ba} \cap \text{Pa}(A_i) = \emptyset$, so no action $A_j \in \text{Ba}$ can affect the action A_i.

Case 3: $A_i \in \text{Ba}$, π_i sentient.

In this case, π_i is (by definition) indifferent as to which act $a_i \in \mathcal{A}_i$ it chooses, regardless of anything else in the game. Therefore the action is simply redundant.

Case 4: $A_i \in \text{Ba}$, $\pi_i \in \Pi_N$.

By definition, the action does not matter to π_i.

Now recall that the BID is common knowledge among the sentient agents in π. So it is reasonable to suppose that the reasoning used above which is based on the BID is also common knowledge. Thus all agents $\pi_j \in \pi \setminus \pi_i$ will ignore an action $A_i \in \text{Ba}$, and treat the game exactly as if A_i did not exist. If π_i is sentient, it will behave in the same manner except that when it comes to take action A_i, it will be indifferent as to the choice of act. And if π_i is an agent of nature, then it is merely the executor of an action which has nothing to do with the rest of the game.

So we are able to remove A_i from the game (delete it and all the arcs connected to it from the BID I to form the new BID I') to leave an equivalent game, in the sense that no other object in the game is affected. In particular, an optimal policy for any agent in a game represented by I will also be optimal in the game represented by I', with the same marginal prospective functions $\{B_i(X|Y) : A_i \in Q,\ \ X, Y \subseteq Q\}$, where $Q = V \cup U \cup A \setminus \text{Ba}$.

Theorem 6.1 (Barren Node Reduction Principle) *A barren node in the BID I of a belief structured game Γ_A may be deleted (along with all its connecting arcs) to form a new BID I' with no effect on the set of non-barren objects $Q = W \cup U \cup A \setminus \text{Ba}$.*

This is a generalisation of a result proved by Shachter (1986) for the decision influence diagram. It is particularly powerful when used in conjunction with arc deletion.

6.1.3 Using arc reversal to remove nodes

As has already been commented on, there is some freedom in the way in which relationships between a set of variables or actions of nature are modelled. So for example, we might have the choice of which direction an arc between two such action nodes will point. Furthermore, under certain conditions, we can reverse the direction of an arc within an existing BID.

This may be accomplished using the arc reversal theorem, postulated by Howard and Matheson (1981) and proved by Shachter (1986) and more generally by Smith (1989b), whose version we give below. Pearl (1988) showed that the arc reversal theorem is a corollary of d-separation (theorem 3.1).

Theorem 6.2 (Arc Reversal) *Let I be an ID which includes the arc (A_i, A_j), where π_i and π_j are both agents of nature. Define I' to be the ID derived from I by:*

(i) removing the arc (A_i, A_j) and replacing it with (A_j, A_i);

(ii) adding the arcs $\{(A_k, A_i) : A_k \in \mathrm{Pa}(A_j) \setminus \mathrm{Pa}(A_i)\}$ and $\{(A_k, A_j) : A_k \in \mathrm{Pa}(A_i) \setminus (\mathrm{Pa}(A_i) \cup A_i)\}$.

Then $C(I) \implies C(I')$.

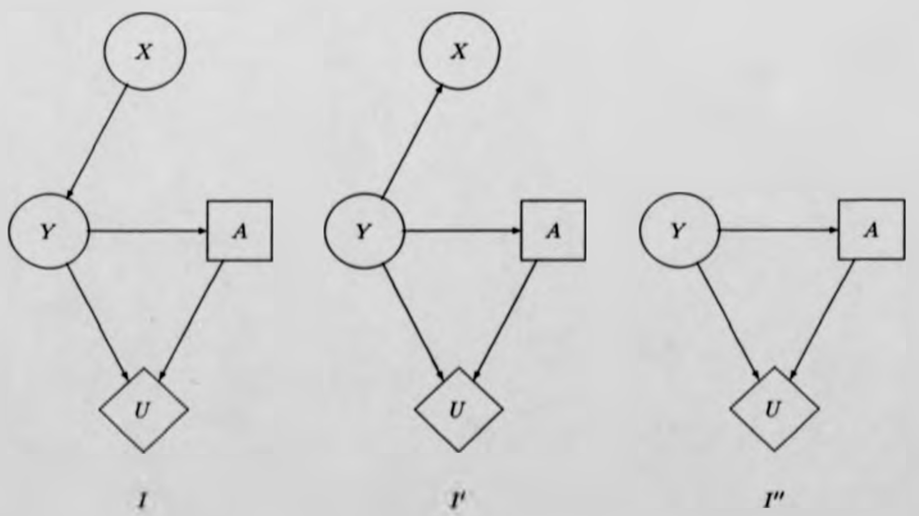


Figure 6.0. Node reduction using arc reversal

If reversing an arc in this way results in a node becoming barren, then it may be removed. So for example, in the ID I shown in figure 6.0, we can reverse the arc between the two actions of nature X and Y to form the ID I'. This leaves X as a barren node, so it can be deleted to leave the ID I''.

6.2 Arc Deletion

6.2.1 The arc deletion theorem

At the end of section 5.1, we stated that the sufficient set $T(A_i)$ is not in general unique. In fact, there may not even be a unique minimal sufficient set $T(A_i)$ — that is such that $T(A_i)$ properly contains no other subset satisfying condition 5.1 — with the result that there may exist two optimal policies neither of which is at least as simple as the other. However, if we restrict the set of belief conditional independence statements to those which can be derived directly from the BID, for example by using the d-separation theorem, the following lemma tells us that there is a unique minimal set $T(A_i)$ satisfying condition 5.1.

Lemma 6.3 (Existence and Uniqueness of the Minimal Sufficient Set) *Let A_i be an action in the game Γ_{A_i} with BID I, such that the b.c.i. statements which hold are exactly those which may be derived directly from I using the criterion of d-separation. Then there exists a unique minimal set $T(A_i)$ and a corresponding maximal set $R(A_i)$ satisfying condition 5.1.*

Proof: Suppose there exist two partitions of $\mathrm{Pa}(A_i)$, $(R(A_i), T(A_i))$ and $(R'(A_i), T'(A_i))$, with $T(A_i) \neq T'(A_i)$.

Define the following subsets:

$$\begin{aligned} W &= R(A_i) \cup R'(A_i), \\ X &= T(A_i) \cap R'(A_i), \\ Y &= R(A_i) \cap T'(A_i), \\ Z &= A_i \cup (T(A_i) \cap T'(A_i)). \end{aligned}$$

So $\{W, X, Y, Z\}$ partition $A_i \cup \mathrm{Pa}(A_i)$.

Now, we recall the proof of the d-separation theorem (theorem 3.1), and form the graph J, which is the undirected, moralised subgraph of $U_i \cup A_i \cup \mathrm{An}(U_i \cup A_i)$, in the same way. Given the premise, the following must hold:

(i) all undirected paths in J between a node $A \in (W \cup X)$ and U_i must pass through a node $B \in (Y \cup Z)$, and

(ii) all undirected paths in J between a node $C \in (W \cup Y)$ and U_i must pass through a node $D \in (X \cup Z)$.

Together, these imply:

all undirected paths in J between a node $E \in (W \cup X \cup Y)$ and U_i must pass through a node $F \in Z$.

Hence, putting

$$\begin{aligned} T''(A_i) &= T(A_i) \cap T'(A_i), \\ R''(A_i) &= R(A_i) \cup R'(A_i), \end{aligned}$$

we have:

$$U_i \perp\!\!\!\perp_B R''(A_i) | A_i \cup T''(A_i) \ ,$$

which contradicts our original assumption.

QED.

We can now prove the Arc Deletion Theorem:

Theorem 6.4 (Arc Deletion) *Let Γ_A be a belief-structured game with BID I, and let $\pi_i \in \pi$ be a sentient agent in Γ_A. If π_i is intelligent, rational and parsimonious, and this is common knowledge among the sentient agents, then the BID I', which is constructed from I by deleting the set of arcs $\{(A_j, A_i) : A_j \in R(A_i)\}$, does not imply any false belief conditional independence statements, where $R(A_i)$ is any subset of the unique maximal set of actions satisfying condition 5.1.*

Proof: Define the following two policy spaces:

$$\begin{aligned} \mathcal{S}_i &= \mathcal{A}_{\mathbf{Pa}(A_i)} \otimes \mathcal{A}_i; \\ \mathcal{T}_i &= \mathcal{A}_{T(A_i)} \otimes \mathcal{A}_i. \end{aligned}$$

By corollary 5.3, if an optimal policy $S_i \in \mathcal{S}_i$ exists (guaranteed by rationality of π_i), then there is an optimal policy $S_i' \in \mathcal{S}_i$ such that

$$S_i'(r, t) = S_i'(r', t)$$

for all $r, r' \in \mathcal{A}_R$, $t \in \mathcal{A}_T$.

Let $\mathcal{T}_i \in \mathcal{T}_i$ be the equivalent policy to $\mathcal{S}_i \in \mathcal{S}_i$. Since the optimal policy $\mathcal{S}'_i$ can be embedded in the policy subspace $\mathcal{T}_i$ then, by proposition 5.4, $\mathcal{T}_i$ is optimal. Similarly, for every optimal policy available to π_i, the equivalent policy in $\mathcal{T}_i$ is optimal.

Now, since π_i is parsimonious, it will employ an optimal policy $\mathcal{T}_i \in \mathcal{T}_i$ which, given the sufficient set $T(A_i)$, does not depend on the set $R(A_i)$. Because this is common knowledge among the sentient agents, we can derive the following belief conditional independence statement:

$$A_i \perp\!\!\!\perp_B R(A_i) | T(A_i) \ .$$

As defined above, the BID I' implies the same b.c.i, statements as does I, together with the above b.c.i. statement and their joint corollaries.

QED.

This is a generalisation of "reduction by sufficiency principle", as given in Smith (1989a).

6.2.2 Using arc deletion

The deletion of an arc in the BID of a game Γ_A is equivalent to an extension of the set of b.c.i. statements which apply to that game. By considering Theorem 6.4 in this way, we can summarise it by the following corollary.

Corollary 6.5 (Arc Deletion Rule) *Let I be the BID for a game, and let it be common knowledge among the sentient agents that π_i is intelligent, rational and parsimonious. Then, for $A_j \in \mathrm{Pa}(A_i)$,*

$$U_i \perp\!\!\!\perp_B A_j | A_i \cup (\mathrm{Pa}(A_i) \setminus A_j) \quad \Longrightarrow \quad A_i \perp\!\!\!\perp_B A_j \,|\, \mathrm{Pa}(A_i) \setminus A_j \ .$$

Proof: Let $U_i \perp\!\!\!\perp_B A_j | A_i \cup (\mathrm{Pa}(A_i) \setminus A_j)$. Then $A_j \in R(A_i)$, according to condition 5.1. Now, by theorem 6.4, deleting the arc (A_j, A_i) from any BID where the above condition holds does not create any false b.c.i. statements.

Hence $A_i \perp\!\!\!\perp_B A_j \,|\, \mathrm{Pa}(A_i) \setminus A_j$, by definition 4.16.

QED.

For example, recall the orientation graph H_1 in figure 4.2. We label the corresponding BID I_1, shown in figure 6.1.

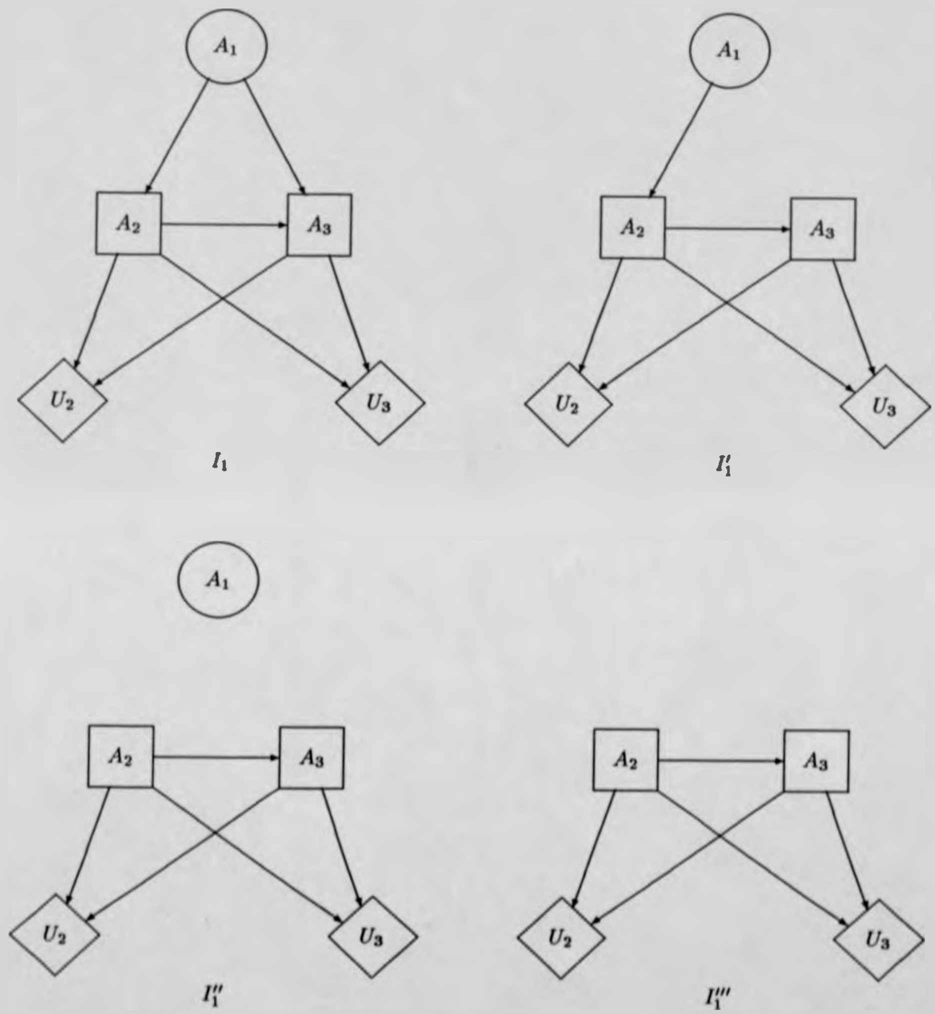


Figure 6.1. Arc deletion and barren node reduction in a BID

We assume that both sentient agents A_2 and A_3 are intelligent, rational and parsimonious, and this fact is also common knowledge to both. Then we can derive the following b.c.i. statement from I_1 using d-separation:

$$U_3 \perp\!\!\!\perp_B A_1 | A_3, A_2 \ .$$

Therefore, by theorem 6.4 we can delete the arc (A_1, A_3) from I to give the BID I_1'.

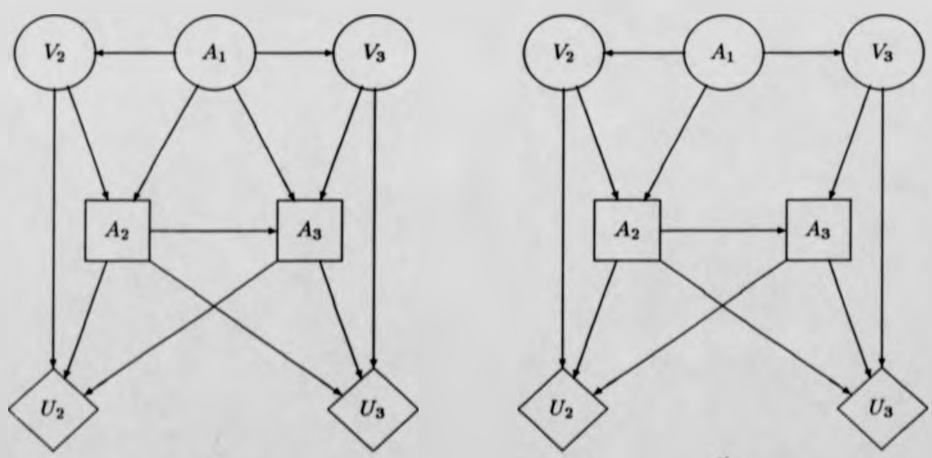
Now, from I_1' we can deduce that

$$U_2 \perp\!\!\!\perp_B A_1 | A_2 \ .$$

So we can now delete the arc (A_1, A_2) to form the BID I_1''.

Finally, from I_1'', we note that A_1 has become a barren node (in this case separated from the rest of the BID), so we may remove it formally using theorem 6.1 to leave the BID I_1'''.

Now consider the alternative representation of the same game as given by the orientation graph H_2 in figure 4.2 and the corresponding BID I_2 in figure 6.2



I_2 I_2'

Figure 6.2. How arc deletion can be restricted by incomplete information

As before, we can delete the arc (A_1, A_3) to form I_2' by noticing that

$$U_3 \perp\!\!\!\perp_B A_1 | A_3, A_2, V_3 \ .$$

However, in this case, we cannot delete the arc (A_1, A_2).

The asymmetry here is due to the fact that π_3 knows A_2, and so has no need to learn anything more about π_2's predisposition V_2 through A_1. In contrast, π_2 does need to take into account A_1, due to the path of influence (A_1, V_3, A_3, U_2). This example also illustrates how incomplete information (as represented in I_2) can affect the qualitative nature of a game.

6.2.3 Uniqueness of the process

We have shown how to derive additional b.c.i. statements, based only on a sentient agent being parsimonious, and thus how to delete arcs in the BID. This process can be successively used on the parent set of every action taken by a parsimonious, sentient agent to further simplify the diagram. (More extensive examples of this are given later in this chapter.) But one question remains. Does it matter in which order the deletions are performed? If the answer is yes then we have a problem, for there will in general be no unique way of reducing the diagram. The following proposition shows that this is not the case.

Proposition 6.6 *Given a set of parsimonious agents $\pi^p \subseteq \pi$, there is a unique minimal BID I' which is obtained by simplifying the BID I using arc deletion as defined above.*

Proof: Given any BID J there is a unique set of b.c.i. statements $C(J)$ which can be derived directly from J (using d-separation). Deleting the arc $\alpha_{ji} = (A_j, A_i)$ from J to form a new BID J' is equivalent to adding the b.c.i. statement $\bar{\alpha}_{ji} = \; A_i \perp\!\!\!\perp_B A_j \,|\, \mathrm{Pa}(A_i) \setminus A_j$ and the joint corollaries of $C(J)$ and $\bar{\alpha}$ (deduced by using the b.c.i. axioms C1–C3) to $C(J)$ to form $C(J')$. Thus we need to show that there is a unique maximal set of b.c.i. statements $C(I')$ which can be derived from $C(I)$ using C1–C3 and the arc deletion rule.

Let $C(I_1)$ and $C(I_2)$ be two maximal sets of b.c.i. statements. Now,

$$C(I) \subsetneq C(I_1)$$

and

$$C(I) \implies C(I_2),$$

therefore

$$C(I_1) \implies C(I_3) := C(I_1) \cup C(I_2).$$

But $C(I_1)$ is maximal, so $C(I_1) = C(I_3)$. Similarly, $C(I_2) = C(I_3)$. Hence they are equal.

QED.

One of the most attractive features of arc deletion using the sufficiency principle is that it only makes use of the BID. I believe such simplicity is a desirable attribute of any methodology. Nevertheless, in practice it is possible to weaken this condition. For the theory to work, we only require the existence of a unique minimal reduction of the BID. It may happen that the introduction of some external information allows for additional simplification without violating this condition. Therefore such a case does not represent any sort of a problem, and can be dealt with in exactly the same way as a simplification based purely on the BID.

6.2.4 Good representation of a game

In order to make arc deletion possible, we have to make important assumptions and restrictions which, although quite sensible, reduce the scope of our results. For instance, we require not only that an agent is rational in the sense of always choosing an optimal policy (the standard normative assumption for a single individual) but that other agents agree about this rationality. We have also artificially restricted our set of valid b.c.i. statements to those which can be derived directly from the BID.

The question of optimality is dealt with in the next section, where we show that a much weaker condition suffices to allow arc deletion. As for the restriction on the set of b.c.i. statements, it is true that there are sets of b.c.i. statements which can not all be represented by a single BID (see for example Smith, 1990). However, it has been argued, for example by Pearl and Verma (1987), that such sets are in some way unnatural, and do not tend to occur in 'real world' situations.

In addition, even if some b.c.i. statements are not represented, we can still obtain a valid set of results by resticting our attention to those which are; it may turn out that following a sequence of simplifications, it may be possible to incorporate some b.c.i. statements which were previously left out. And on a practical note, the d-separation theorem has only been proved for a set of b.c.i. statements which can be represented in a BID, so the fact that some are not represented may not reduce the number of deductions and simplifications we can make.

To illustrate what might happen, let us consider a case where there is an additional b.c.i. statement not implied by the diagram.

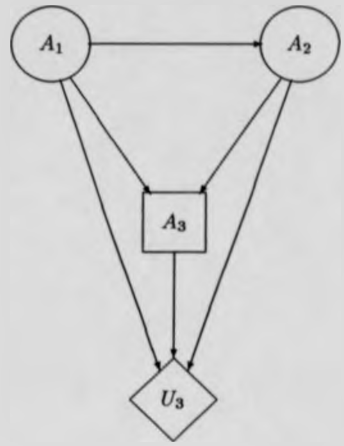


Figure 6.3. The BID of a poorly-modelled game

Clearly in figure 6.3, $C(I) = \emptyset$. Now suppose the action and utility spaces are:

$$\mathcal{U}_3 = \mathcal{A}_i = \{0, 1\} \text{ for all } i,$$

with $A_1 = A_2$, and

$$U_3 = \left(A_3 + \tfrac{1}{2}(A_1 + A_2)\right) \pmod 2.$$

Then give any distribution satisfying the above constraints, we can derive the following two b.c.i. statements:

$$U_3 \perp\!\!\!\perp_B A_1 | A_2, A_3 \ ;$$

$$U_3 \perp\!\!\!\perp_B A_2 | A_1, A_3 \ .$$

The arc deletion rule now gives:

$$A_3 \perp\!\!\!\perp_B A_1 | A_2 \ ;$$

$$A_3 \perp\!\!\!\perp_B A_2 | A_1 \ .$$

But there is no unique reduction, so the graph cannot be simplified.

So what has gone wrong? Although I is not a false model of the game, it is a poor one because it puts in two nodes (A_1 and A_2) where one will do. Note that of the two b.c.i. statements implied by the actual distribution, one but not both could be represented in the BID. I would guess that most examples in which we have a set of b.c.i. statements which cannot be included in a single BID represent a poor model of a game, similar to the example above; thus restricting ourselves to the BID may be a positive move, in the sense that it rules out a large class of 'poor' models.

6.3 Extensions

In this section we will look at some ways of extending the principle of arc deletion, by varying the extent to which agents can be described as rational. There are two directions in which we can go. By strengthening the rationality conditions to include a concept of joint or mutual rationality we can perform further deletions in a BID. But first we consider a weakening of the rationality conditions, namely dropping the assumption of optimality.

6.3.1 Dropping the optimality assumption

Suppose we require that every agent is parsimonious, in other words that parsimony is accepted as an axiom. We believe such an axiom is not unreasonable, given the arguments set out in chapter 5. However the consequences of this are quite profound. It means that the simplification of the BID is based solely on the graph itself, without any reference to 'external' factors, such as the distribution of random objects associated with the game.

Now, in the influence diagram literature, many methods of simplifying an ID have been developed. (For an extensive list see Smith, 1988.) Arc deletion via sufficiency is one of the few which rely exclusively on the graph.

If we follow the above line of reasoning to its natural conclusion, we see that optimality is not necessary for arc deletion to take place. This may sound counter-intuitive, but recall that the graph is defined according to the b.c.i. property, which corresponds not to *optimality*, but to *indifference* on the part of the agent. To see how this works, we need only to prove the sufficiency principle without the assumption of optimality.

Theorem 6.7 (Generalised Sufficiency Principle) *Let A_i be the action taken by a sentient agent π_i, and let there exist sets $R(A_i)$ and $T(A_i)$ satisfying condition 5.1.*

Suppose there exists a policy $S_i \in \mathcal{S}_i$. Then there is a policy $S_i' \in \mathcal{S}_i$, with $S_i' \sim S_i$, such *that,*

$$S_i'(r,t) = S_i'(r',t)$$

for all $r, r' \in \mathcal{A}_R, \quad t \in \mathcal{A}_T$.

Proof: The only point at which optimality is required for the Sufficiency Principle is in the proof of theorem 5.2. Now suppose some act $a_i^* \in \mathcal{A}_i$ is not optimal given experience (r,t) under our original partial preference ordering $\mathcal{P}(r,t)$. Then define a new partial preference ordering $\mathcal{Q}(r,t)$ which is a function of $\mathcal{P}$ such that:

(i) $a_i \sim_{\mathcal{Q}} a_i' \iff a_i \sim_{\mathcal{P}} a_i'$ for all $a_i, a_i' \in \mathcal{A}_i$

(ii) $a_i^* \succsim_{\mathcal{Q}} a_i'$ for all $a_i' \in \mathcal{A}_i$.

Under $\mathcal{Q}$, there is no change in the set $C(I)$ of b.c.i. statements, so condition 5.1 still holds. Hence the proof of theorem 5.2 follows without the assumption of optimality, and we are done.

QED.

Corollary 6.8 (Generalised Arc Deletion) *If the sentient agent* π_i *is parsimonious, and this is common knowledge among the sentient agents, then the BID I as defined in Theorem 6.4 does not imply any false b.c.i. statements.*

Proof: Direct from theorems 6.4 and 6.7.

QED.

6.3.2 Joint parsimony

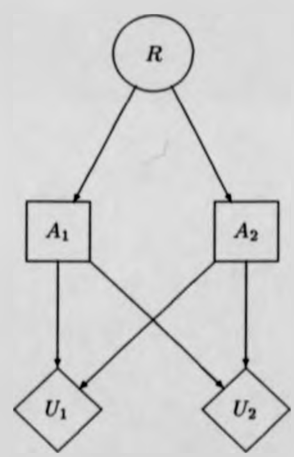
As was mentioned earlier in this chapter, an action known to several agents may be removed in some cases, depending on the rationality of the agents concerned. We now consider what type of rationality may be used to justify this.

Consider the effects of including additional dependencies in decision-making. In chapter 5, we noted that parsimony has no effect on the outcome of a game. Essentially this is because the discarded information has no direct implications for the utility, and is either unavailable to other agents, or (applying repeated arc deletion) is assumed to be ignored by those agents to which it is available.

However, in many cases, an action which would otherwise be deemed irrelevant may be known to more than one agent. Using such information to determine which act to take may inform other agents about what the agent in question might do. This may be either beneficial, inconsequential, or harmful to the agent concerned.

The three examples we consider here all have a similar pattern. In each case, we have two sentient actors, Π_1, Π_2, and nature Π_N, each with one agent (π_1, π_2, and π_3 respectively). $R = A_3$ represents the state of the weather, either 'rain' or 'no rain'. Knowing R, both sentient agents take their actions A_1 and A_2 simultaneously, with both payoffs U_1 and U_2 depending only on A_1 and A_2, that is,

$$C(I) = \ U_i \perp\!\!\!\perp_B R | A_1 \cup A_2 \quad \text{for } i = 1, 2.$$



I_3

Figure 6.4. Common knowledge of R

All three games can be represented by the same BID I, as shown in figure 6.4. Note that the predisposition nodes are not included in this BID This is because in each of the examples we consider, the utilities are represented in a normal-form matrix. Therefore the utility of each actor is a function of the outcome which is common knowledge to all. In other words, all predispositions are common knowledge, so they can be suppressed in the model, according to the conditions set out above.

First consider the well-documented game 'battle of the sexes'. A husband Π_1 and wife Π_2 have to choose whether to go to a football match or to an opera, but are unable to communicate with each other what their choice is. The husband prefers football and the wife prefers the opera, but neither wants to go to an event on their own. The normal form utility matrix for this game is shown in figure 6.5.

		Π_2	
		football	opera
Π_1	football	(2,1)	(0,0)
	opera	(0,0)	(1,2)

Figure 6.5. Normal-form utilities for 'battle of the sexes'

Clearly, if both agents decide to go for their favourite event, both will lose out. And if they try to anticipate the other's action, there is still a substantial possibility that they gain nothing. So suppose they have previously decided that if it rains, they will go to the opera, and if it doesn't they will go to the football. During the day, they both observe an identical value of R.

Now, this information (the agreement together with the observation) has no direct bearing on the utilities; in this model we assume that the football is indoors. However, it clearly does have a bearing on what the other will do, and it can be used to the advantage of both. The 'solution' described is an example of a correlated equilibrium (see for example Aumann 1974, 1987), with R being the correlating mechanism.

Secondly, consider another well-known game, 'paper, scissors, stone'. Two agents simultaneously choose either 'paper', 'scissors', or 'stone'. Paper beats stone, stone beats scissors, scissors beats paper, and the same choice produces a tie. The normal form utility matrix for this game is shown in figure 6.6.

Now suppose it starts to rain. Clearly, this should make no difference to the behaviour of the agents. However, one or both agents might for some reason choose to let their optimal policy depend on whether it rained or not. Provided the other agent does not know in what way this difference in behaviour manifests itself, for instance if they play a one-off game, then no harm is done.

Π_2

Π_1

	paper	scissors	stone
paper	(1,1)	(0,2)	(2,0)
scissors	(2,0)	(1,1)	(0,2)
stone	(0,2)	(2,0)	(1,1)

Figure 6.6. Normal-form utilities for 'paper, scissors, stone'

However, suppose one agent does have some information about the other's idiosynchrasy. For instance, it might decide never to choose 'paper' when it rains, 'in case the paper gets wet'. In that case, its opponent would be given an unnecessary oportunity to gain an advantage.

Clearly such a policy makes no sense at all, and could be described as 'irrational'. However, under the current definition, parsimony is insufficient to preclude such a possibility. In fact, to do so, we need a much stronger version of the parsimony principle, involving not only common knowledge of parsimony, but a common agreement on the irrelevance of information.

Although the examples above are somewhat contrived, it turns out that in real life such cases are the norm. In practice, there will always be seemingly irrelevant pieces of information which are common knowledge to some of the agents in a game. The question of deciding which are indeed irrelevant goes right to the heart of any attempt to devise a closed-model approximation to an open system, which is the essence of statistical model building. Therefore it deserves some comment.

How do the agents in a game jointly decide to ignore certain information? The answer is partly systematic or rational, and partly to do with culture or tradition. First, consider the thought processes of an actor playing the game, 'paper, scissors, stone'. He thinks:

> 'Is there any reason why the rain should change the way I play? Not unless it affects my opponent. And I think she's pretty sensible, so is unlikely to be affected. So

I may as well assume it makes no difference, unless my experience of playing the game tells me otherwise. I choose to ignore the rain since it involves negligible loss of information.'

In other words, he applies the *scientific method*. And providing his opponent does the same, we can remove the factor of rain from our model.

As an example of the effect of culture on a game, consider the game of bridge. During a game, a player may recieve 'unauthorised' information due to a mistake by his partner. It is considered unethical for that player to use this information to his advantage. Therefore all players will make the assumption that this infomation is ignored by the player in question.

We can encompass examples such as these within our system by using the concept of joint parsimony.

Definition 6.2 *Say the set of agents $\pi^* \subseteq \pi$ are* jointly parsimonious with respect to *an arc (A_j, A_i) if it is common knowledge among π^* that π_i ignores A_j when taking A_i.*

Result 6.9 *If $\pi^* = \Pi_S$, we can delete the arc (A_j, A_i) from the BID.*

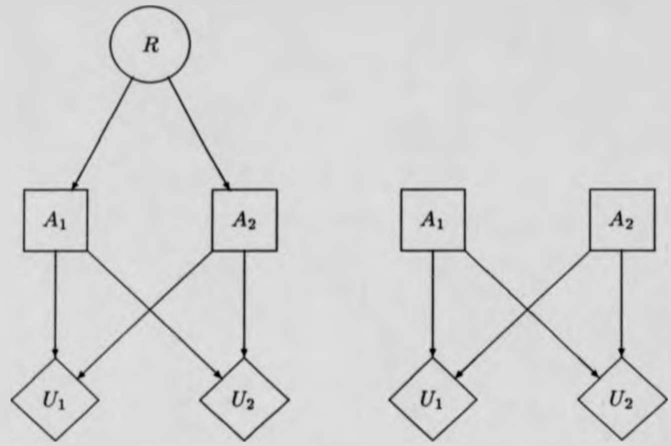
6.3.3 Equilibral reductions

The above definition for joint parsimony serves merely to describe a phenomenon: namely that an agent chooses to ignore some action for which it knows the value. It does not address the question as to why it acts in this way; in particular we might ask whether it is 'rational' for it to do so. Now we have already seen that such joint parsimony is justified given the conditions required for the sufficiency principle. Are there any other circumstances in which it would be reasonable to expect a 'rational' agent or agents to disregard previous actions in this way?

Smith and Allard (1992) employ an equilibrium-based argument to derive a principle known as equilibral rationality. Let I be the BID of the belief structured game Γ_A. Let X be a subset of the set of actions taken by sentient agents. For each $A_i \in X$,let $R^*(A_i)$ be a subset of $\mathrm{Pa}(A_i)$. Now let I^* be the BID obtained from I by deleting the set of arcs $\{(A_j, A_i) : A_i \in X, A_j \in R^*(A_i)\}$.

Consider the BID I' derived from I^* by replacing one arc (A_j, A_i) which has been removed from the BID I. If applying the arc deletion rule to I' would result in the arc (A_j, A_i) being deleted again, then we say that I^* is a partial equilibral reduction of I with respect to (A_j, A_i).

Definition 6.3 *We say I^* is an* equilibral reduction *of I if it is a partial equilibral reduction with respect to (A_j, A_i) for all $A_i \in X$ and all $A_j \in R^*(A_i)$.*



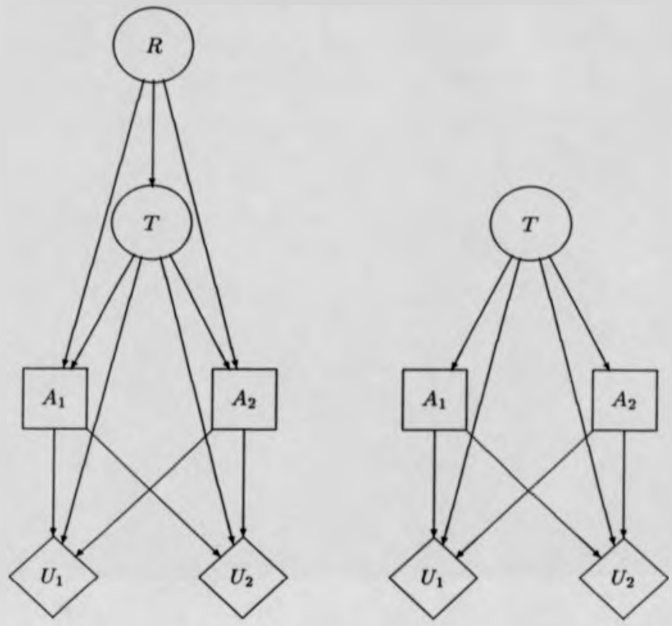
I_3 $\qquad\qquad\qquad$ I_3^*

Figure 6.7. Reduction using equilibral rationality

Consider the BID I_3 shown in figure 6.7. Clearly, if the arc (R, A_1) were not there, then we could remove the arc (R, A_2) via arc deletion, and vice-versa. Therefore removing both arcs gives an equilibral reduction I_3^*, shown in figure 6.7 with the barren node R removed.

A more complex example is depicted in figure 6.8. The arcs (R, A_1) and (R, A_2) may both be deleted from the BID I_4 to form an equilibral reduction I_4^*, in which the node R has been removed following reversal of the arc (R, T).

Equilibral rationality requires all sentient agents simultaneously to ignore certain previous acts. The principle behind it is that it is parimonious (and optimal) for every agent to ignore certain actions provided every other agent does likewise: hence our use of the word equilibral. In practice it shows that in a large number of situations, the use of extended arc deletion using joint parsimony may be justifiable.



I_4 $\qquad\qquad$ I_4^*

Figure 6.8. Another example of reduction

Equilibral reductions are typically non-unique. Indeed, when co-operation or co-ordination is mutually advantageous, as in a correlated equilibrium, then such a reduction might be termed (jointly) 'irrational'. Equilibral reduction may remove from the game some information, commonly known by two or more players which acts as a correlating mechanism, as in the 'battle of the sexes' described above.

Maskin and Tirole (1989) define a Markov perfect equilibrium by assuming independence of all subsequent actions in the extensive form, given the actions already taken; see also Fudenberg and Tirole (1991). It turns out that this corresponds to a special case of equilibral reduction, namely the minimal equilibral reduction $I^{\min}$. We define $I^{\min}$ to be the result of removing the maximum number of arcs possible; if the arc deletions are restricted to those allowed according to the BID, it is unique by the proof of proposition 6.6. Needless to say, Markov equilibrium is a not uncontroversial concept in game theory.

However, in a positive sense, the set of equilibral reductions provides us with a nested set of possible models or hypotheses for a game, against which we may be able to compare what actually happens. Any BID which is not an equilibral reduction may represent a model of a game which is untenable in the strong normative sense. In other words, we would not be able to recommend the optimal policies arising from it simultaneously to all agents (which does not mean to say that it can't happen).

For example, Young and Smith (1992) consider a symmetric repeated game involving two actors, Π_1 and Π_2. They show that if the actor Π_1 assumes that her opponent Π_2 is using a strategy (known as an n-step back strategy) based only on the latest n pairs of actions, then she can improve on Π_2's performance by using all previous actions. Thus in the Bayesian setting it is proved that strategies based on the lastest n pairs of moves cannot form an equilibral reduction.

6.4 An Example

To end this chapter, we will look at an example of how a game can be modelled and simplified using the BID methodology we have developed. We use a game adapted from Parthasarathy and Raghavan (1971).

6.4.1 A game similar to poker

This is a zero-sum game with two sentient actors Π_1 and Π_2, and we assume that the utility of each actor (and hence each agent) is measured in units. Each is dealt randomly and without replacement one of three cards labelled 1, 2 and 3, and we denote the card dealt to actor Π_i as X_i. However, before either actor has seen his card, Π_2 can choose whether or not to switch the two cards, in the action A_2^1 taken by agent π_2^1. The card which is finally allotted to and seen by actor Π_i is denoted Y_i.

The minimum bet or 'ante' is 1 unit. Π_1's first action A_1^1 (taken by agent π_1^1) is either to 'raise' the bet to 2 units, or to 'pass' and leave it at 1. Then Π_2's agent π_2^2 takes A_2^2, which is a choice between 'raise' and 'pass' if A_1^1 was 'pass', and a choice between 'see' and 'fold' if A_1^1 was 'raise'. Finally, if he 'passed' and Π_2 'raised', then Π_1 has another action A_1^2 which is either to 'see' or 'fold'.

If both pass, then the actor with the higher numbered card wins 1 unit. If one raises and the other sees, then the actor with the higher numbered card wins 2 units. If one actor folds, then the other wins 1 unit. By identifying 'pass' and 'fold' with the act L and 'raise' and 'see' with the act H, we see that each of the actions A_1^1, A_1^2 and A_2^2 has the same action space $\mathcal{A} = \{L, H\}$.

We define Z to be a binary response variable (action of nature), taking the value 1 if $Y_1 > Y_2$ and 0 otherwise. To simplify the diagram, we denote X to be the ordered pair (X_1, X_2), U to be the set of utilities $\{U_i^j\}$ and V the set of predispositions V_i^j, where

$$U_1^1 = U_1^2 = -U_2^1 = -U_2^2$$

and V is common knowledge, since we assume that the utility function is common knowledge to both actors. We also assume perfect recall on the part of both actors.

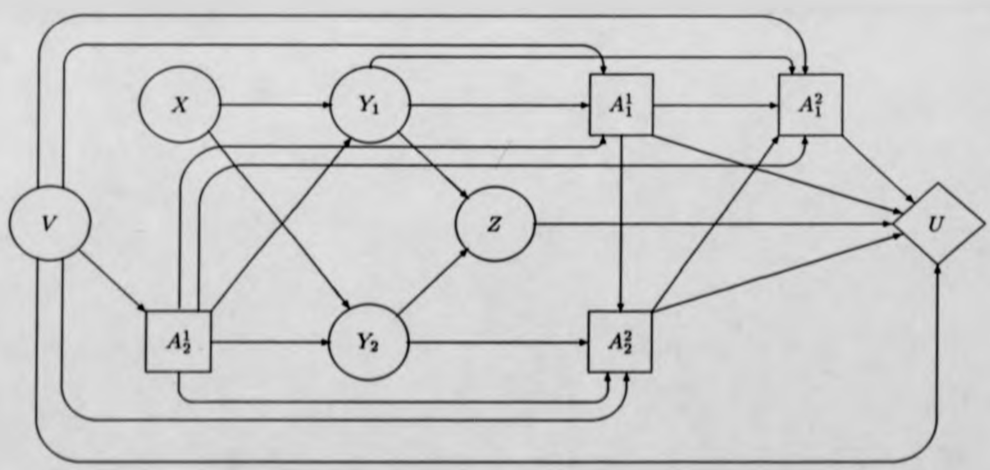


Figure 6.9. The complete BID for the poker-like game

The complete BID is shown in figure 6.9. Although we have already simplified it by using the vectors X, U and V, it still contains 10 nodes and 24 directed arcs. By using all of the techniques for simplification developed here combined with some general principles of statistical modelling, and making the appropriate assumptions at each step, we see how the BID may be reduced to only four nodes and four arcs.

Step 1. The first thing to note is that the predisposition vector V is common knowledge to all sentient agents. So it does not need to be explicitly represented in the BID, and may be removed along with its connecting arcs according to the arguments set out in section 6.1.1.

Step 2. Now suppose that each agent's prior prospective function for X, namely $B_i^j(X|\emptyset)$, is symmetrical with respect to each of the six possible values. Then since π_2^1 does not admit X, we must have $Y_i \perp\!\!\!\perp_B A_2^1$ given any subset of $\mathrm{Pa}(A_i^j)$, $i, j = 1, 2$. Thus we may delete the arcs (A_2^1, Y_i).

This results in the BID shown in figure 6.10.

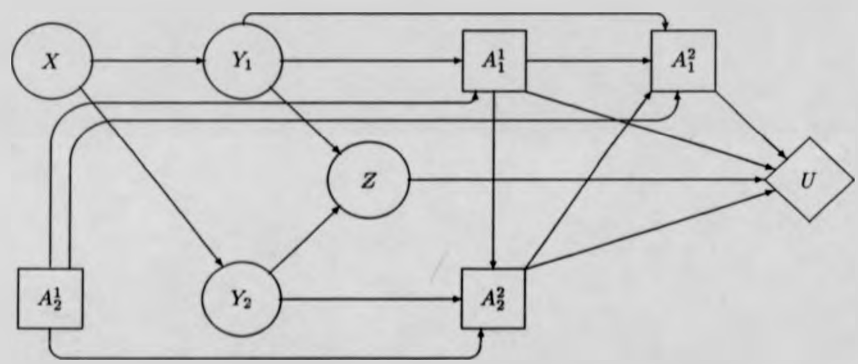


Figure 6.10.

Step 3. By theorem 6.2 we can reverse the arc (X, Y_1), and then subsequently the arc (X, Y_2), adding the arc (Y_1, Y_2).

Step 4. As a result, X is now a barren node, so may be deleted with its connecting arcs according to theorem 6.1.

Step 5. Now, suppose that π_1^2 is intelligent and parsimonious. Using d-separation (theorem 3.1) we may deduce the b.c.i. statement,

$$U \perp\!\!\!\perp_B A_2^1 | A_1^2, A_1^1, A_2^2, Y_1.$$

By the arc deletion rule (corollary 6.5) we have,

$$A_1^2 \perp\!\!\!\perp_B A_2^1 | A_1^1, A_2^2, Y_1 \ ,$$

and we may delete the arc (A_2^1, A_1^2).

Step 6. This gives us

$$U \perp\!\!\!\perp_B A_2^1 | A_1^1, A_2^2, Y_2 \ ,$$

so the arc deletion rule allows us to delete (A_2^1, A_2^2). This in turn leads to

$$U \perp\!\!\!\perp_B A_2^1 | A_1^1, Y_1 \ ,$$

and the deletion of the arc (A_2^1, A_1^1).

Step 7. This leaves us with the disconnected node A_2^1, which is formally removed as a barren node, to produce the BID in figure 6.11.

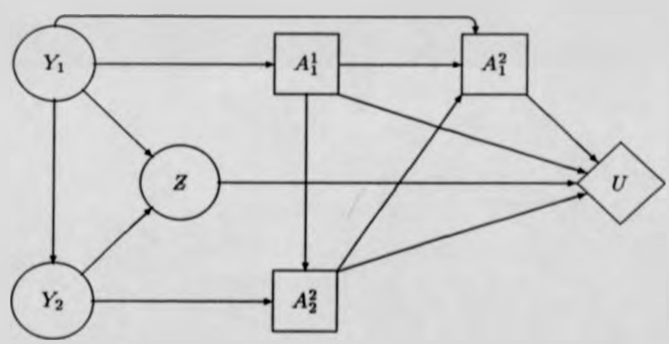


Figure 6.11.

Step 8. Recall that the action A_1^2 is only ever taken if $a_1^1 = L$ and $a_2^2 = H$, so if π_1^2 has to take A_1^2 then these two acts may be assumed. Strictly speaking, we may suppose A_1^2 is always taken but that it only ever makes any difference to U when the aforementioned acts are chosen.

Therefore by the strong independence principle, or the 'sure thing' principle of Savage (1954), it is parsimonious for π_1^2 to ignore A_1^1 and A_2^2, and let its act a_1^2 depend only on Y_1. In other words,

$$A_1^2 \perp\!\!\!\perp_B A_1^1, A_2^2 | Y_1 \ ,$$

and we may delete the arcs (A_1^1, A_1^2) and (A_2^2, A_1^2).

Step 9. Now this BID represents any game with the same qualitative structure. So for example the numbers on the cards, instead of being restricted to 1, 2 and 3, could have any distribution. Suppose it is common knowledge to all sentient agents that (X_1, X_2) — and hence (Y_1, Y_2) — are independently distributed on $\Re$. Then we have $Y_2 \perp\!\!\!\perp_B Y_1$, and we can delete the arc (Y_1, Y_2).

The BID is now reduced to the one shown in figure 6.12.

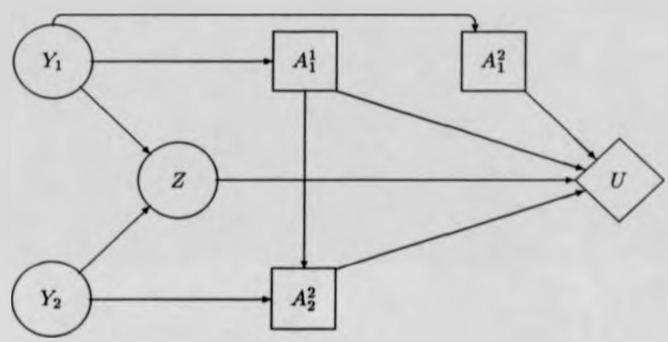


Figure 6.12.

Step 10. Finally, let us consider what we can deduce if each sentient agent has a prior prospective function $B_i^j(\cdot|\emptyset)$ such that Y_i is uninformative about Z. This would happen, for example, if the prior distribution was a suitable improper distribution, such as Y_i ~independent uniform on $\Re$. If this is common knowledge, we have $Z \perp\!\!\!\perp_B Y_i$, and so (Y_i, Z) is deleted, for $i = 1, 2$.

Step 11. Consequently $Z \perp\!\!\!\perp_B A_i^j$ for all i, j, and we may remove Z, according to the conditions set down in section 6.1.1.

Step 12. Noticing that $U \perp\!\!\!\perp_B Y_2 | A_2^2$, (Y_2, A_2^2) can be got rid of by the arc deletion rule.

Step 13. Now, suppose that the arc (Y_1, A_1^1) wasn't there. Then we would have $U \perp\!\!\!\perp_B Y_1 | A\ _1^2$, and the arc (Y_1, A_1^2) could be deleted. Conversely if (Y_1, A_1^2) did not exist then (Y_1, A_1^1) would go. This satisfies the conditions for an equilibral reduction, according to definition 6.3, and so we may be able to delete both arcs simultaneously.

Step 14. The nodes Y_1 and Y_2 have become barren, so may be removed to leave the final simplified BID depicted in figure 6.13.

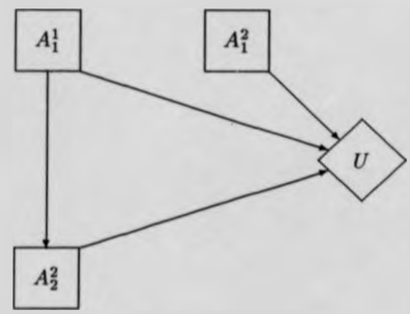


Figure 6.13.

6.4.2 Discussion

Of course, the above example cannot be considered a serious analysis; it is primarily an exercise, designed to illustrate how a BID might be simplified, given a variety of assumptions. In fact, we have managed to illustrate every method for simplifying a BID which was described in this chapter. Perhaps more importantly, we have shown how these methods can alternate and interact with general modelling principles and the introduction of specific distributional characteristics to produce some quite powerful results.

On another level, we have illustrated the enormous flexibility of the BID representation compared to the extensive form. The extensive form of a game allows you to vary the utilities and not much else. Adding or removing an action means drawing a new tree. One BID allows you to vary the utilities, the action spaces and sometimes the order of actions. Adding or removing actions is relatively easy, and the qualitative implications are straightforward to calculate.

By performing manipulations on a BID, we can evolve a whole family of possible game structures. Most importantly, we can analyse the relationships between those structures, and specify *explicitly* what sets of assumptions are necessary and/or sufficient for two structures to represent the same game. Also, we have illustrated the power of some of the seemingly obvious assumptions we made during the analysis. For anyone considering such assumptions, the consequences, as seen by the rapid disintegration of the BID, would soon dissuade them from believing them to be entirely innocuous.

And once again it is worth emphasising that none of the analysis relies on the use of specific distributions. Hence the inferences made will be valid for any distribution which falls within the equivalence class of games represented by that single BID, and of which the simplified poker game is but one member.

Of course, actually to *solve* a game — whatever that means — you have to introduce a lot more quantitative information. But there are still advantages in performing some preliminary analysis using BIDs. It facilitates the simplification of a game, prior to introducing specific distributions, resulting in economy of effort and a reduction in computation. Such considerations become more important when dealing with games of greater complexity. And significantly it informs on the structure of the data, creating a better understanding of the nature of a game.

7 Repeated Games

7.1 Modelling Finite Repeated Games

While we have shown that a BID can be used to model just about any finite game, it is not necessarily the case that the resulting BID can usefully be simplified using only the methods developed in the previous chapter. In particular, in the case of a game for which the BID turns out to be very dense (meaning relatively many pairs of nodes have directed arcs between them), the initial set of deducible b.c.i. statements will be that much smaller; hence the conditions for simplification will not occur to any great extent, if at all.

Thus it appears that, once a certain level of complexity is reached, the BID can do little more than provide a graphical representation of a game. While that is a very useful first step in analysing a game, it may do little more than confirm just how difficult it would be to undertake a more comprehensive analysis.

But the BID is very useful in modelling a game with large numbers of actions, provided that the resulting graph is relatively sparse. With a repeated game this will often be the case. Furthermore, a BID representing a repeated game will exhibit repeated patterns and symmetries. It turns out that we can take advantage of this relatively formal graphical structure.

7.1.1 Schematic BIDs

We recall from section 2.3.3 that a repeated game comprises a number of consecutive plays of a stage game, each stage game being identical; so the same *actors* would take actions in the same order, each action having the same action space as its counterpart in every other stage game. As in chapter 4, and for the same reasons, we restrict our attention to finite games and assume that the BID is common knowledge. As an illustration of a finite repeated game, The BID of a repeated prisoners dilemma game with T stages is shown in figure 7.1.

As we do throughout this chapter, to simplify the appearance of the BID, a schematic representation is used. A_i^j is the action of *actor* Π_i at stage j, taken by his agent π_i^j. H^0 represents the situation at the start of the game: in this case just the predispositions of the two actors. H^j is the situation or *history* immediately after stage j has been completed, and includes the initial state H^0 and all the previous actions A_i^j. U_i is the utility rewarded to actor Π_i, and hence to each of his agents π_i^j.

Although the BID is imprecise, in the sense that it has been simplified for the purposes of representing a repeated game, it still displays the properties normally associated with any BID. In particular, the (imprecise) b.c.i. statements which can be derived from it using d-separation, for example, are still valid.

In the game shown in figure 7.1 we have assumed complete information, so that the predispositions of both actors are common knowledge. We have also assumed perfect recall, so that all previous actions and hence the entire history is common knowledge to the agents participating in any given stage.

However, this need not be the case. We can use the BID to model a game of incomplete information without perfect recall. This is done by partitioning H^{j-1} according to what is admitted by each agent participating in stage j. Thus if there are n such agents, for each one we must partition H^{j-1} according to what it knows. We then take the join of all n partitions to give us the appropriate partition of H^{j-1} for the BID.

Since in general this would result in 2^n nodes in place of each history node H^j we will, for the sake of simplicity, confine our discussion principally to games with two actors. However, the generalisation to games with any finite number of actors, including multiple actions by an actor within a stage and actions by nature, is straightforward.

Figure 7.2 shows a generalised BID for a stage game with two sentient agents. The two action nodes A_i^t are unconnected, although this need not be the case. H^{t-1} is partitioned into four subsets. K^{t-1} represents the part which is common knowledge to both agents; I_i^{t-1} represents π_i's private information; and J^{t-1} is unknown by either agent.

Note that in both these examples, H^j will always be a complete summary of everything which has gone before. So it is trivially sufficient, in both the usual statistical sense and that of chapter 5. Thus the representation of perfect recall does not require that the parental set of one action is a subset of the parental sets of an actor's subsequent actions. Instead, we have all the information repeated between each stage. While this representation is initially used for convenience, it has some important consequences as we will discover later on.

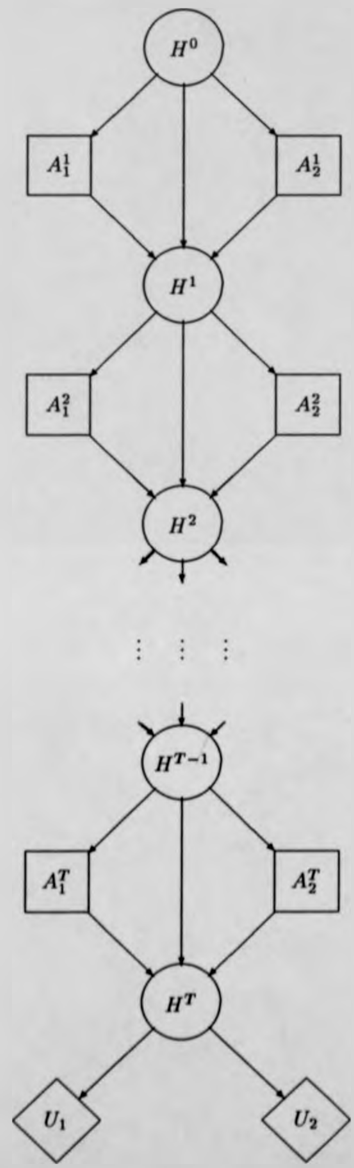


Figure 7.1 A BID Representation of a Repeated PDG

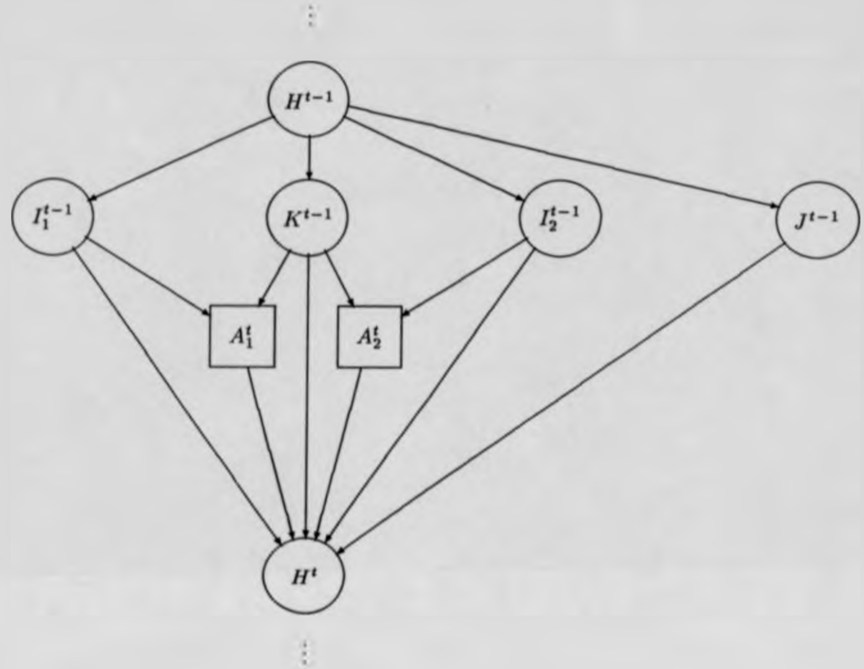


Figure 7.2 The BID for a generalised stage game with two sentient agents

7.1.2 Learning in a Repeated Game

Having made the link between agents belonging to an actor, we can now consider how that actor learns during the course of a game. To start with we will confine our attention to the type of repeated game discussed above, in which an actor takes only one action in each stage. This provides us with a natural (chronological) ordering of his actions, and of the agents which take those actions.

The obvious starting point is to consider the increase in the knowledge of an actor from one stage to the next, as represented by the difference between what is admitted by two of his agents in consecutive stages. So a measure of learning by actor Π_i between stage $t-1$ and stage t might be the difference between $\mathrm{Pa}(A_i^t)$ and $\mathrm{Pa}(A_i^{t-1})$. But there is no reason why we should restrict our definition to consecutive actions, or even to an ordered pair of actions. This suggests that the difference between $\mathrm{Pa}(A_i^k)$ and $\mathrm{Pa}(A_i^j)$ might be used to measure learning.

So in our first example, the repeated PDG shown figure 7.1, since the complete predisposition (in other words the normal form) is common knowledge, learning is confined simply to which acts have been chosen. In the general case (as shown in figure 7.2) the learning by actor Π_i between stages j and k might be measured by the difference between $K^k \cup I_i^k$ and $K^j \cup I_i^j$.

Such a definition can encompass games in which an actor has more than one action per stage. It can also relate to the forgetting of information (or negative learning) in cases where perfect recall is not assumed. It can also be used to describe learning in any game, whether or not it exhibits the structure of a repeated game.

But learning is not simply a matter of acquiring (or losing) knowledge about the values of certain actions. Associated with such changes in certain knowledge are changes in the beliefs an actor holds (through his agents) about actions the values of which are unknown. In other words, learning should include any change (or updating) which takes place from the prospective function of one agent to that of the next, conditioned on that change in knowledge.

If an agent is intelligent, then we can represent its knowledge as a subset of its prospective function. Therefore, for an intelligent actor — defined to be an actor all of whose agents are intelligent — there is a natural extension to the definition of learning outlined above.

Definition 7.1 *Let Π_i be an intelligent actor, and let A_i^j and A_i^k be actions taken by the agents π_i^j and π_i^k, both belonging to Π_i. We define* learning $L(\Pi_i)_j^k$ *by actor Π_i to be the change from $B_i^j(\cdot \mid \mathrm{Pa}(A_i^j))$ to $B_i^k(\cdot \mid \mathrm{Pa}(A_i^k))$, where $B_i^h(\cdot \mid \cdot)$ is the prospective function of agent π_i^h.*

If the actor is a Bayesian, then the prospective functions of his agents will be conditional probability statements, with learning occurring through the usual procedure for updating and conforming to the standard coherence conditions. An actor employing Dempster-Shafer Belief functions could use Dempster's (1968) algorithm for combining evidence. And no doubt there are 'sensible' ways of updating beliefs for almost any type of prospective function.

But our definition does not require that the learning which takes place necessarily follows any such rule. Given that agents may act independently, even if they belong to the same actor, then their prospective functions may also be unrelated. Hence the learning which takes place may as a result appear haphazard to an external observer, even though each agent may be *individually* 'rational'.

7.1.3 Locally optimal policies

While playing in a game over a long period of time, it is often difficult to think of the maximisation of some far-off utility function as a realistic goal. Instead an actor might try to maximise some local measure of how well he is doing in the game. The usual method for representing this in a repeated game is to have some payoff result from each stage, and then allow the final utility to be some monotone increasing function of the set of payoffs; the most common models involve either the sum or discounted sum of stage payoffs.

Our system of modelling uses no such implicit device. In the case of a game in which money is won and lost at each stage, the amount of money held at the end of any stage may be taken as a measure of success, but this is not true in general; in any case such a measure, even if it exists, may not be a particularly good one. Indeed the same applies to the payoff history except under very limited conditions. For an actor must take account of how gains now may lead to losses later on, due to the future reactions of opponents, as in the repeated PDG.

But there is an alternative local measure of success which we can employ. For any stage j, we have defined H^j to be the complete history of the game at the end of that stage. Therefore H^j would seem to be an obvious candidate. Now we recall from definition 5.2 that,

> a sentient agent π_i^j prefers experience $H^j = h^j$ to experience $H^j = h^{j\prime}$ (written as $h^j \succ h^{j\prime}$) if
>
> $$B_i^j(U_i|h^j) > B_i^j(U_i|h^{j\prime}).$$

Thus we have a local measure of each agent's success in a game up to any given point. But can an agent use this measure as a goal on which to focus, as an alternative to the final utility? There is no guarantee that an act which an agent believes will result in a preferred experience H is the same as an optimal act. However, if there is a certainty that a preferred experience will result from such an act, whatever else happens, then that act will be optimal.

Definition 7.2 *Suppose H^j can be defined as a function of the sets of actions X^j and Y^j. Then we say that a sentient agent π_i^j weakly prefers experience x^j to $x^{j\prime}$* locally, with respect to H^j *($x^j \succsim_{H^j} x^{j\prime}$) if*

$$H^j|x^j \cup y^j \succsim H^j|x^{j\prime} \cup y^j$$

for all $y^j \in \mathcal{A}_{Y^j}$.

In other words, one possible experience is locally preferred to another if it is, in the words of Savage (1954), a 'sure thing' that whatever else obtains, one will lead to a preferred history.

Definition 7.3 *Given experience* x^j, *we define an act* $a_i^{j*} \in \mathcal{A}_i^j$ *to be* locally optimal with respect to H^j *if for all* $a_i^{j\prime} \in \mathcal{A}_i^j$,

$$(x^j \cup a_i^{j*}) \underset{H^j}{\succsim} (x^j \cup a_i^{j\prime}).$$

And we can define locally optimal strategies accordingly. While a locally optimal act may not always exist, if it does then it is always optimal, as the next theorem shows.

Theorem 7.1 *A locally optimal act is optimal.*

Proof: Suppose a_i^{j*} is locally optimal with respect to H^j given experience $x^j \in \mathcal{A}_{X^j}$. And let $Y^j = \mathrm{Pa}(H^j)\backslash(X^j \cup A_i^j)$. Then, assuming that all the terms used below are well-defined, we have

$$\begin{array}{rll} & (x^j \cup a_i^{j*}) & \underset{H^j}{\succsim} (x^j \cup a_i^{j\prime}) \text{ for all } a_i^{j\prime} \in \mathcal{A}_i^j \\ \Longrightarrow & H^j | a_i^{j*} \cup x^j \cup y^j & \succsim H^j | a_i^{j\prime} \cup x^j \cup y^j \text{ for all } y^j \in \mathcal{A}_{Y^j} \\ \Longrightarrow & B_i^j(U_i | a_i^{j*} \cup x^j \cup y^j) & \succsim B_i^j(U_i | a_i^{j\prime} \cup x^j \cup y^j) \\ \Longrightarrow & B_i^j(U_i | a_i^{j*} \cup x^j) & \succsim B_i^j(U_i | a_i^{j\prime} \cup x^j) \\ \Longrightarrow & (a_i^{j*} \cup x^j) & \succsim (a_i^{j\prime} \cup x^j) \\ \Longrightarrow & a_i^{j*} & \succsim a_i^{j\prime} \end{array}$$

hence A_i^{j*} is optimal.

QED.

So if an agent can find a policy which is locally optimal, then that policy must also be optimal. It may happen in a particular game that such locally optimal policies are easier to determine than (globally) optimal policies.

7.1.4 The Local Sufficiency Principle

Returning to the main theme of this chapter, we now make use of the schematic BID representation to establish a further method which can be used to simplify a repeated game.

Recall that in both the examples we considered in section 7.1.1, the nodes H^j could be seen to separate one stage game from the next. This is shown clearly in figure 7.3, in which the set of actions taking place in each stage is represented by a single node A^j, and the set of utilities is denoted by a single node U. The BID should in this case be read as if it were a DAG, with all the nodes representing random objects. So in particular, the directed arc from H^{j-1} to A^j does *not* imply that H^{j-1} is common knowledge to every sentient agent participating in stage j.

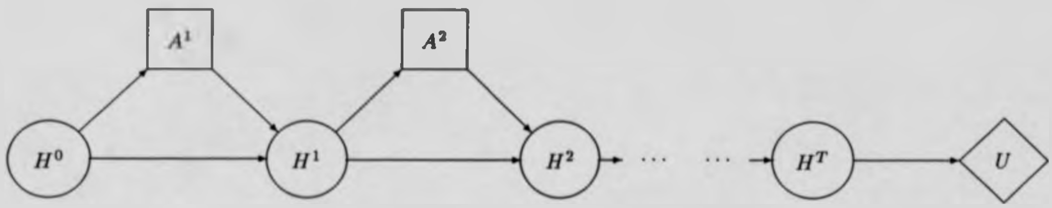


Figure 7.3 A BID representing the separation of stages by history nodes

The BID shown in figure 7.3 has the following Markov properties:

$$\bigcup_{j=0}^{t-1} H^j \perp\!\!\!\perp_B \bigcup_{j=t+1}^{T} H^j | H^t \cup X \ . \tag{7.1}$$

$$Y \perp\!\!\!\perp_B U | H^T \cup X \ . \tag{7.2}$$

for $t = 1, 2, \ldots, T-1$, $X, Y \subseteq Q$.

Since that schematic BID can be used to represent every repeated game, these properties can be said to apply generally. Furthermore we may without loss of generality denote A^j to be the set of actions taken after H^{j-1} and before H^j.

We can now state and prove the following theorem.

Theorem 7.2 (Local Sufficiency Principle) *Let Γ_A be a repeated game with Markov properties (7.1) and (7.2). Suppose for some $A_i^j \in A^j$ taken by the sentient agent π_i^j, whose utility is denoted by U_i, that*

$$H^j \perp\!\!\!\perp_B R(A_i^j) | A_i^j \cup T(A_i^j) \ , \tag{7.3}$$

where $R(A_i^j)$ and $T(A_i^j)$ partition $\mathrm{Pa}(A_i^j)$.

And suppose there exists an optimal policy $\bar{S}_i^j \in S_i^j$. Then there is an optimal policy $\tilde{S}_i^j \in S_i^j$ such that

$$\tilde{S}_i^j(r,t) = \tilde{S}_i^j(r',t)$$

for all $r, r' \in \mathcal{A}_R$, $t \in \mathcal{A}_T$.

Proof: By the sufficiency principle (corollary 5.3) it is sufficient to show that,

$$U_i \perp\!\!\!\perp_B R(A_i^j) | A_i^j \cup T(A_i^j) \ . \qquad (*)$$

Using the c.i. axioms C2 and C3, we have:

$$(7.1) \implies R(A_i^j) \perp\!\!\!\perp_B H^T | H^j \cup A_i^j \cup T(A_i^j) \qquad (7.4)$$

$$(7.2) \implies R(A_i^j) \perp\!\!\!\perp_B U_i | H^T \cup H^j \cup A_i^j \cup T(A_i^j) \qquad (7.5)$$

$$(7.4),(7.5) \implies R(A_i^j) \perp\!\!\!\perp_B U_i | H^j \cup A_i^j \cup T(A_i^j) \qquad (7.6)$$

$$(7.3),(7.6) \implies R(A_i^j) \perp\!\!\!\perp_B U_i | A_i^j \cup T(A_i^j) \ . \qquad (*)$$

QED.

Corollary 7.3 *If π_i^j is parsimonious, and this is common knowledge among the sentient agents, then the arcs $(R(A_i^j), A_i^j)$ may be deleted from the BID without implying any false b.c.i. statements.*

Proof: Direct from Theorems 6.4 and 7.2.

QED.

The significance of this theorem is that it allows us to make deductions based only on a small segment of the BID, in this case relating to a single stage. In fact, provided we can find a set of nodes which play the same role as H^j, separating part of the BID from everything which follows it, then we can make the same sort of deductions.

The problem with the theorem is that, as currently defined, H^j is a summary of *everything* which precedes it in the game. Therefore the set $R(A_i^j)$ will be empty. The solution to this problem is to find some sufficient subset (if possible a minimal sufficient subset) of H^j, so that the deductions to be made from the local sufficiency principle are no longer trivial. In the sense we use it here, sufficiency relates specifically to the agent's utility.

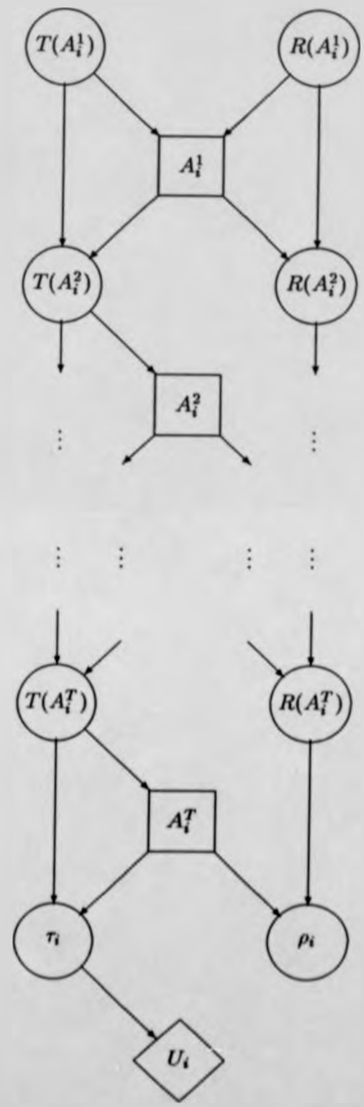


Figure 7.4 Using the local sufficiency principle

Consider the schematic BID shown in figure 7.4. This is drawn for the just the single actor Π_i, with the nodes $R(A_i^j)$ and $T(A_i^j)$ playing the role of H^{j-1}. We see that the nodes ρ_i and $R(A_i^j)$, for $j = 2, \ldots, T$ can be deleted as barren nodes, and that $R((A_i^1)$ can subsequently be deleted using the local sufficiency principle. While there is nothing particularly noteworthy about this, the next example takes the reasoning a step further.

7.2 Examples

We now consider two examples which illustrate the capabilities and limitations of this new local method of simplification.

7.2.1 A proof in game theory using BIDs

Our first example is adapted from Smith (1988). A repeated game with T stages is played between two actors, Π_1 and Π_2. In each stage j which is played, Π_1 and Π_2 simultaneously take actions A_1^j and A_2^j, respectively. Each such pair of actions results in a payoff to Π_1 which is a function of a_1^j and a_2^j, known only by Π_1.

We assume that Π_1 is parsimonious. We say an actor Π_i is parsimonious if, for all $\pi_i^j, \in \Pi_i$, it is common knowledge to $\{\pi_i^k : \pi_i^k \in \Pi_i\}$ that π_i^j is parsimonious.

Being parsimonious, Π_1 needs to find a strategy (a set of policies, one for each of his agents) such that each policy depends on some set of actions which is minimal with respect to his utility U_1. In doing so, he must condition on his beliefs about Π_2, which are represented by his prospective function for the actions to be taken by her agents. In our model, we will assume that A_2^j depends only on the past through the statistic τ_2^{j-1}. We also assume that U_1 is monotone increasing in Π_1's payoff, aggregated over the whole game, where Σ_j represents his aggregate payoff after the jth stage.

The BID in figure 7.5 shows the complete structure of the game.

In this case, the role of H^j is taken by the three nodes, Σ^j, R^j and τ_2^j. The simplification is not immediate as it was in the last example. Instead we must go by backwards induction. Consider Π_1's last action A_1^T. Setting

$$T(A_1^T) = \Sigma^{T-1} \cup \tau_2^{T-1},$$

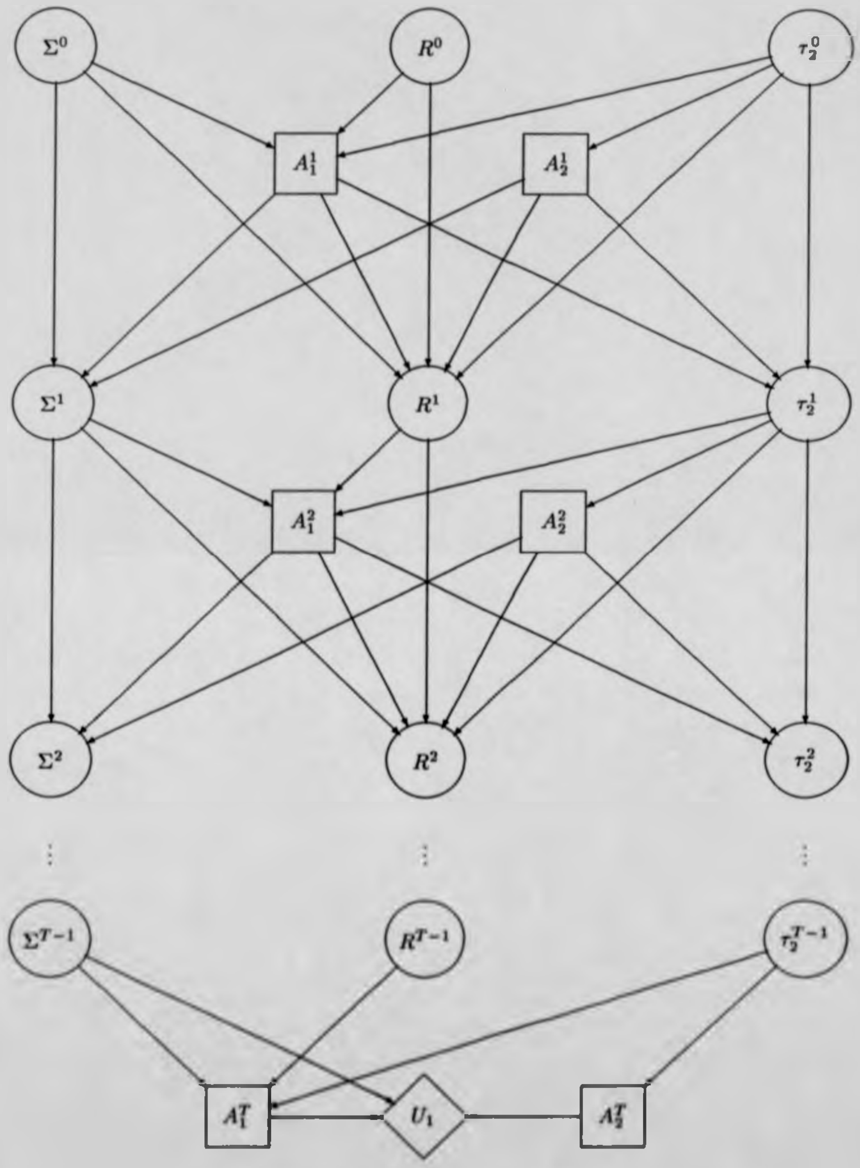


Figure 7.5 The BID for example 7.2.1

we see that condition (5.1) from lemma 5.1 is satisfied, so that provided π_1^T is parsimonious, the arc (R^{T-1}, A_1^T) may be deleted using generalised arc deletion (corollary 6.8).

Now if we consider Π_1's penultimate action A_1^{T-1}, we find that the arc (R^{T-2}, A_1^{T-1}) may be deleted via local arc deletion (corollary 7.2), making use of similar reasoning. And in general, if agents $(\pi_1^i, \ldots, \pi_1^T)$ are all parsimonious, then having already deleted the arcs $(R^{j-1}, A_1^j)_{j=i+1}^T$, we will be able to delete the arc (R^{i-1}, A_1^i) using local arc deletion.

Now we know that the *actor* Π_1 is parsimonious, so that all his agents are parsimonious. Then, by induction, we can delete all arcs $(R^{j-1}, A_1^j)_{j=1}^T$, to produce the reduced BID shown in figure 7.6.

Thus we have proved the following proposition.

Proposition 7.4 *Given the model of a repeated game as described above, then for any policy of the parsimonious actor* Π_1 *at stage* j*, there exists a policy which depends on the past only through the statistic* τ_2^{j-1} *and his aggregated payoff* Σ^{j-1}*, such that* Π_1 *is indifferent between the two policies.*

This is a generalisation of the theorem given by Smith (1988), who restricted the analysis to the case in which Π_1 is a Bayesian, and so aims to find an *optimal* strategy through maximisation of expected utility. The version given above includes not only the Bayesian as a special case, but any other criterion for optimal behaviour, and even 'non-optimal' behaviour.

As with all the other analysis using BIDs, we have a result which both defines the required conditions more explicitly and can be easily generalised. The main qualification we have added to the usual Bayesian version of this result is that the actor must make the assumption that he will in future act in a parsimonious way, in order for the sufficiency condition to hold.

The proof used is essentially rigorous, and does not involve any algebraic equations. It is therefore straightforward to gereralise it to any case which fits the stated constraints. For example, if Σ^j is any function of only Σ^{j-1}, A_1^j and A_2^j then the result still follows. So we could allow U_i to be any explicit function of the set of stage payoffs, for instance a function of any discounted aggregate payoff.

In our final example, we incrementally develop a more complicated model, and demonstrate how increasing complexity eventually renders our simplification methods ineffective.

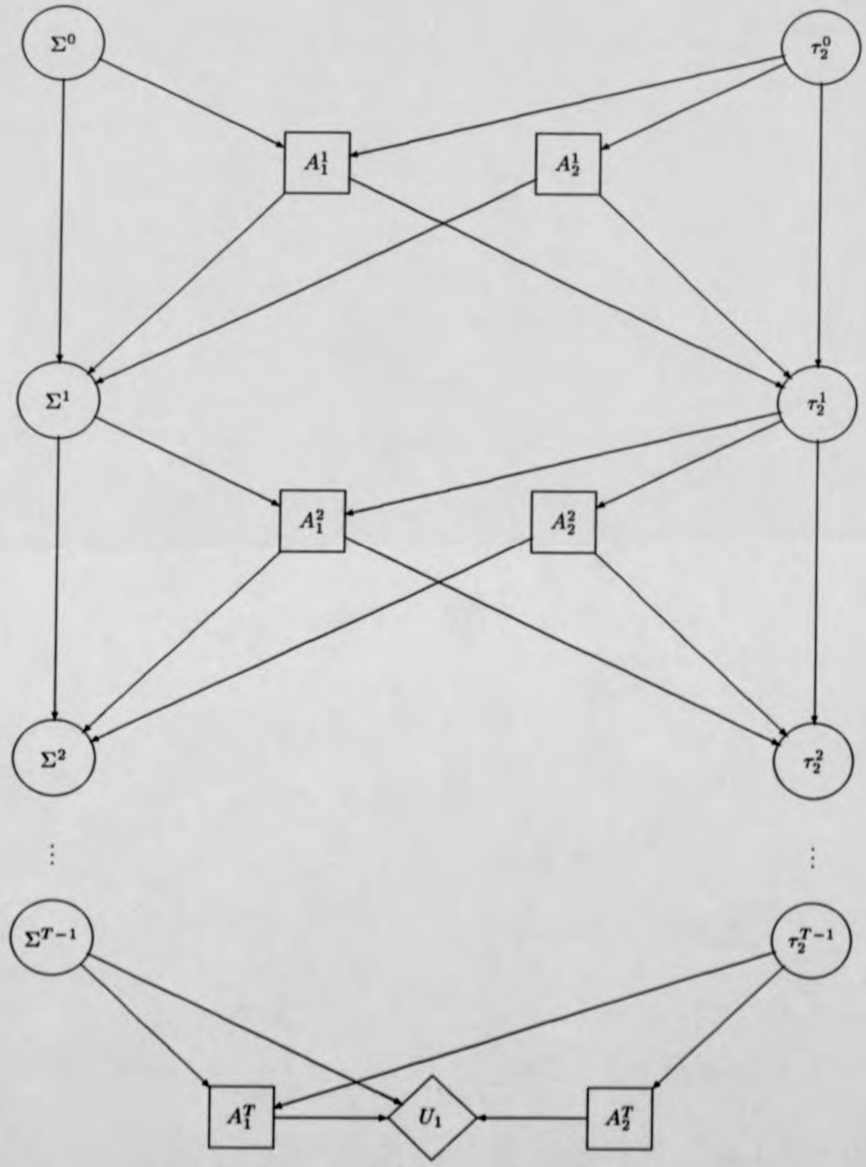


Figure 7.6 The reduced BID from example 7.2.1

7.2.2 Modelling a competitive market

This example is adapted from Smith and Allard (1992).

Consider a market supplied by three companies, Π_1, Π_2, and Π_3, whose sales in month t are denoted by X_1^t, X_2^t, and X_3^t respectively. Company Π_2 is the market leader, and attempts to sell to every customer. Π_1 supplies an 'upmarket' brand, while Π_3 caters for the 'downmarket' end. There is a population (assumed to be fixed) consisting of two types of consumer, type a and type b. Type a consumers choose between the products of companies Π_1 and Π_2 only, while type b choose between those of Π_2 and Π_3. This is an example of a partially-segmented market, as modelled using DAGs by Queen (1994).

The sales of Π_2's product in month t to type a and type b consumers are respectively $X_2^t(a)$ and $X_2^t(b)$. However, these values are never observed directly but only their sum $X_2^t = X_2^t(a) + X_2^t(b)$. A simple influence diagram for these variables for the month t is given in figure 7.7

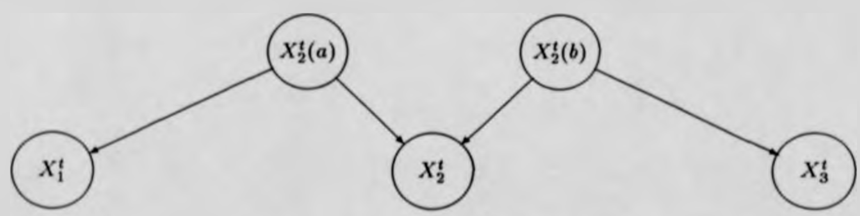


Figure 7.7 The market in month t

For the sake of argument, we will suppose that the utility U_i of company Π_E depends only on the actions taken after some arbitrarily chosen month, setting $t = 0$ for that month.

Before every month each company can choose whether or not to pay for additional advertising to increase its sales at the expense of its competitor(s). Company Π_2 as market leader has to act early in order to preserve its competitive advantage, and so in month 1 has only its own sales figures X_2^0 on which to base its action A_2^1. In contrast, Π_1 and Π_3 wait to see what Π_2 does before taking their own actions A_1^1 and A_3^1. In the meantime, the monthly sales figures for all three companies $X^0 = (X_1^0, X_2^0, X_3^0)$ are published. So both know X^0 as well as A_2^1 when

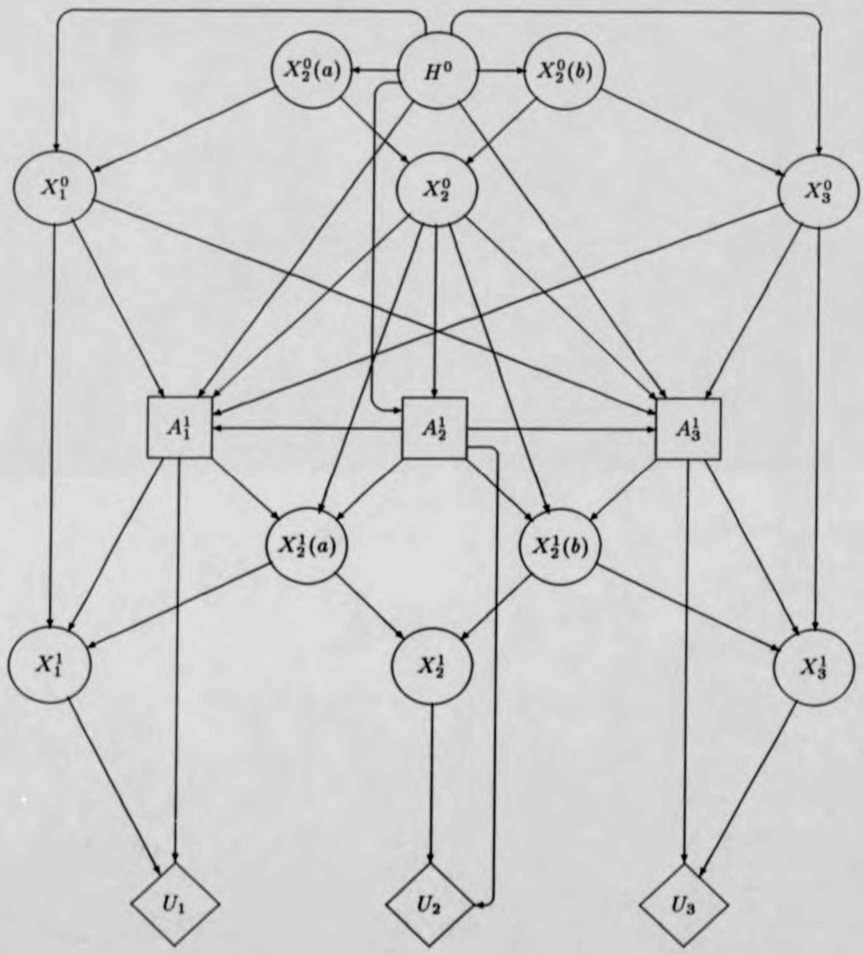


Figure 7.8 The BID of a game played on a partially segmented market

they come to choose their actions.

Discussions with Π_1's market research director suggest that the distribution of $X_2^1(a)$ depends on (X_2^0, A_1^1, A_2^1) only, and a similar model is proposed for $X_2^1(b)$. Since we are working within a partial-segmentation model, we can assume that A_2^1 only affects the competitors' sales X_1^1 and X_3^1 via its effects on $X_2^1(a)$ and $X_2^1(b)$, and that the actions taken by Π_1 and Π_3 have no effect on the other's market segment. Thus we postulate that the distributions of X_1^1 and X_3^1 depend only on $(X_1^0, A_1^1, X_2^1(a))$ and $(X_3^0, A_3^1, X_2^1(b))$ respectively.

Now suppose that U_i represents the first month's profits for company Π_i. So U_i depends only on A_i^1 and X_i^1. The complete BID for this game is shown in figure 7.8. The history node H^0 represents the set $\{X^t, A^{t+1} : t < 0\}$, where $A^t = (A_1^t, A_2^t, A_3^t)$.

From this BID, we can deduce the b.c.i. relation,

$$U_1 \perp\!\!\!\perp_B X_3^0, H^0 | A_1^1, X_1^0, X_2^0, A_2^1 \ .$$

Therefore the arcs (X_3^0, A_1^1) and (H^0, A_1^1) may be deleted if it is common knowledge that Π_1 is parsimonious. Similarly the arcs (X_1^0, A_3^1) and (H^0, A_3^1) may be deleted.

But now consider what happens when we move the time horizon forward by just one month. The utility U_i of each company is now a function of the four actions $(A_i^1, A_i^2, X_i^1, X_i^2)$. The complete BID for the two-stage game is shown in figure 7.9.

For the agents participating in the final stage (stage 2), we can make similar deductions to those made for the one-stage version. We have

$$U_1 \perp\!\!\!\perp_B X_3^1 | A_1^2, X_1^1, X_2^1, A_2^2, H^1 \ .$$

So the arc (X_3^1, A_1^2) — and by similar reasoning the arc (X_1^1, A_3^2) — may be deleted.

In this case the arcs (H^1, A_1^2) and (H^1, A_3^2) cannot be deleted as the BID stands. However we may choose to replace them with the arcs (A_1^1, A_1^2) and (A_3^1, A_3^2), since for actors Π_1 and Π_3, their previous action is a sufficient statistic for the game's history with respect to their utility.

But when it comes to considering simplificaions in stage 1, a different picture emerges. The b.c.i relation

$$U_1 \perp\!\!\!\perp_B X_3^0, H^0 | A_1^1, X_1^0, X_2^0, A_2^1$$

is no longer true (in general).

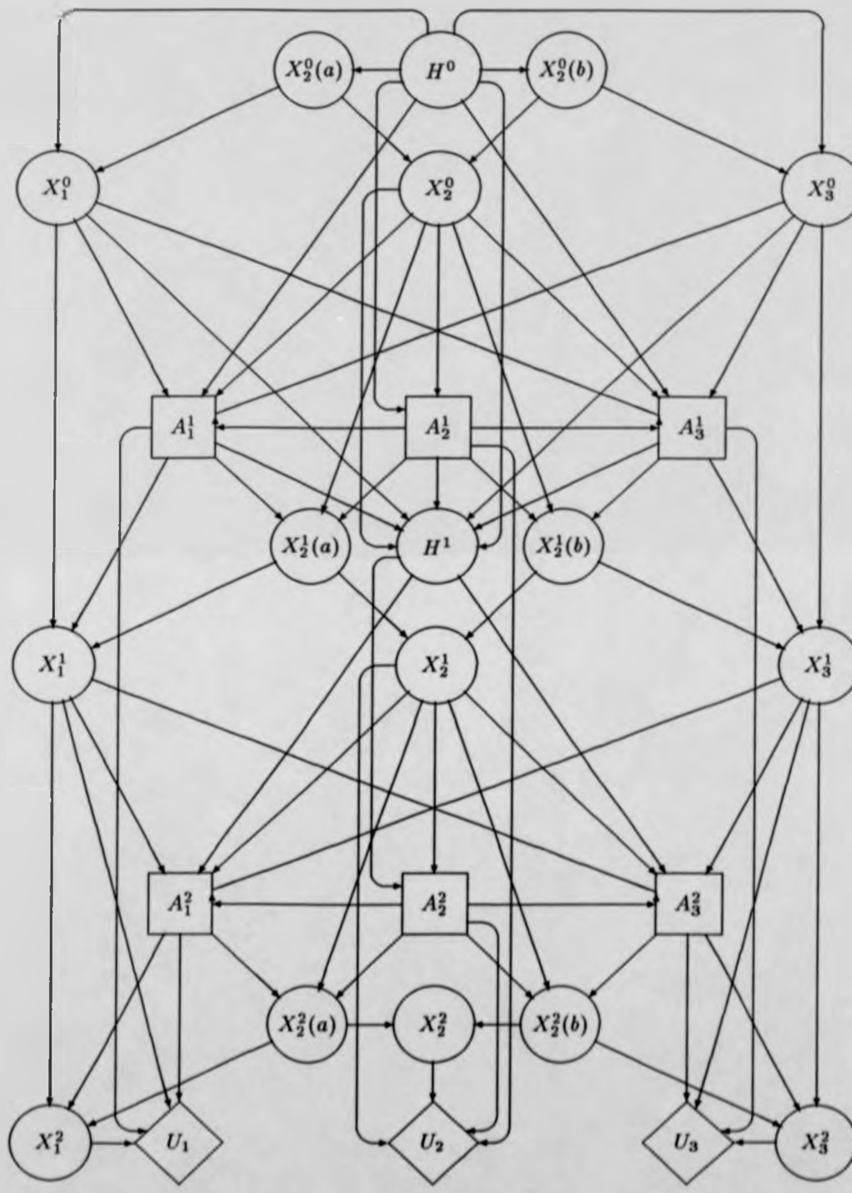


Figure 7.9 Second order effects in a competitive market

To see why, notice that in figure 7.9, there is now a path from X_3^0 to U_1 which does not pass through the set $A_1^1 \cup \mathrm{Pa}(A_1^1)$, which was not the case in the 1-stage game shown in figure 7.8. For example, consider the path $(X_3^0, A_3^1, X_2^1(b), X_2^1, A_2^2, X_2^2(a), X_1^2, U_1)$. From here, we see that the action taken by company Π_3 in stage 1 can have an effect on the sales of company Π_1 in stage 2 via the reaction by company Π_2 in stage 2 to the results of Pi_3's first action. Hence due to this 'second-order' effect, Pi_1 can no longer afford to ignore the sales figures X_3^0 of company Π_3, even though they are not directly competing with each other.

Similarly, we can not immediately delete the arcs (H^0, A_1^1) and (H^0, A_3^1). However, if the required conditions hold, those two arcs may be removed using equilibral rationality or some other form of joint parsimony.

Thus we are prevented from deducing very many simplifications directly from the BID due to the sheer complexity of the game. As we can see from figure 7.9, the BID is more dense (it has a larger number of connected node pairs) than any we have considered so far. However, the model is still potentially very useful. In a real-life situation, it is possible that some of the arcs may be absent or may represent very weak dependencies. These can be tested using data, as can the resulting simplifications. It is this aspect of modelling which we consider in the last section.

7.3 Qualitative Hypothesis Testing

Consider the market game represented by the BID in figure 7.8 from the point of view of an individual who is outside the market, but who has some interest in it, for instance a market regulator, or a consumer. (Although consumers as a whole can influence the market, we presume that an individual cannot, so is not properly an actor in our model.) He will only know those variables which are in the public domain, namely the sales figures, (X^0, X^1). Nevertheless, if we assume that the BID is objectively accurate and that Π_1 is parsimonious,he can immediately deduce from our qualitative model the following relation on those variables,

$$X_1^1 \perp\!\!\!\perp_P X_3^0 | X_1^0, X_2^0 ,$$

which was not previously apparent. An analogous argument based on parsimony by Π_3 gives,

$$X_3^1 \perp\!\!\!\perp_P X_1^0 | X_3^0, X_2^0 ,$$

To take things a little further, a company could measure or estimate most of the variables in the model (using market research), and compare the historic data with the above model. Typically, additional conditional independence statements might be inferred from the data. (For a practical example of this process, see Wermuth and Lauritzen, 1990.)

If the same analysis can be done by all three companies (for example if the relevant historic data was in the public domain), then such c.i. relations might be assumed to be common knowledge, and so could be represented by deleting the appropriate arcs form the BID. We might then find that the model could be simplified again, just by re-applying the sufficiency principle.

It is worth noting that in practice this will often not be the case, since a company's internal decisions are generally kept secret from its competitors. However, the inferences to be drawn based on any assumption can be tested against the data, and can throw some light on whether that assumption may be valid.

Suppose, for example, that we wish to test the hypothesis that the only effect of additional advertising is to induce consumers to switch brands, and that it has no effect on the overall size of the market. This would be represented in the BID in figure 7.9 by the deletion of the arcs $\{(A_i^t, X_i^t) : i = 1,3\}$. From the BID, we can immediately see that (in the case of company Π_1) we need to verify the statement,

$$X_1^t \perp\!\!\!\perp_P A_1^t | X_1^{t-1}, X_2^t(a) \ .$$

Since $X_2^t(a)$ is not observable, we would have to estimate it for each month t, for example through market research; once we start having to estimate such quantities, any deductions which result are no longer logical, but have some degree of uncertainty attached. Thus one conclusion which can be drawn is that to determine the effect of advertising in a partially segmented market would require additional quantitative assumptions about the relationships between variables in that market.

Alternatively, suppose that company Π_1 claims that it does know quite a lot about the market: that the number of type a consumers is a commonly known constant α, and that they have no brand loyalty. Using the 1-stage model of figure 7.8, this analysis would suggest the deletion of the arcs (X_1^0, X_1^1) and $(X_2^0, X_2^1(a))$.

As a consequence, we may deduce that Π_1 can ignore both X_1^0 and X_2^0 when taking action A_1^1. We might ask whether the company is really prepared to ignore its previous month's sales figures, and those of its competitor, when deciding the advertising policy. If Π_1 allows its action to be affected by those figures, then we must deduce that either the company is behaving irrationally or it does not really believe the stated assumptions about the type a consumers.

If, however, Π_1 does ignore those sales figures then, since we have already shown that H^0 and X_3^0 can be ignored, the only action in the BID it uses to help decide A_1^1 is the action A_2^1 of its competitor. Now suppose that company Π_3 is able to make similar assumptions about its potential customers of type b. This leaves us with the BID shown in figure 7.10.

Using the sufficiency principle, we can now delete the arcs (H^0, A_2^1) and (X_2^0, A_2^1) to give us the result,

$$X^1 \perp\!\!\!\perp_P X^0 \ .$$

Whether or not the various assumptions made above are sensible is not the point. What we have shown is that the BID provides an excellent framework against which to test many qualitative assumptions and hypotheses. And we need not restrict such hypothesis testing to consideration of data. We can take into account the nature of decision making within a company or by an individual actor.

But most importantly, the BID allows us to deduce the consequences of any hypothesis relatively easily and to test those consequences against the data as well. To see the advantage a BID provides for this type of analysis we need only look back to figure 7.9. While the BID is quite large, it is easy to isolate the dependencies which relate to an individual node.

Thus we can see that a practical application for BIDs is the building and evaluation of models of competitive markets and other similar situations. The consideration of a variety of BIDs might be used to assist a company in answering the following types of questions:

Is the model appropriate *for our purposes*; can we simplify it, or should we be including some additional factors? Are the decisions we take based on the right sort of information? What information should we be paying more attention to and what can we afford to ignore; is there some information we don't have which is useful enough to spend more money in acquiring it?

What factors influence the decision making processes of our competitors; based on past performance, how predictable and/or rational is their behaviour? How might they react to our decisions, and which particular factors will they pay most attention to? And finally, is the model supported by historical data; can we use the BID to help derive further relationships between variables based on that data?

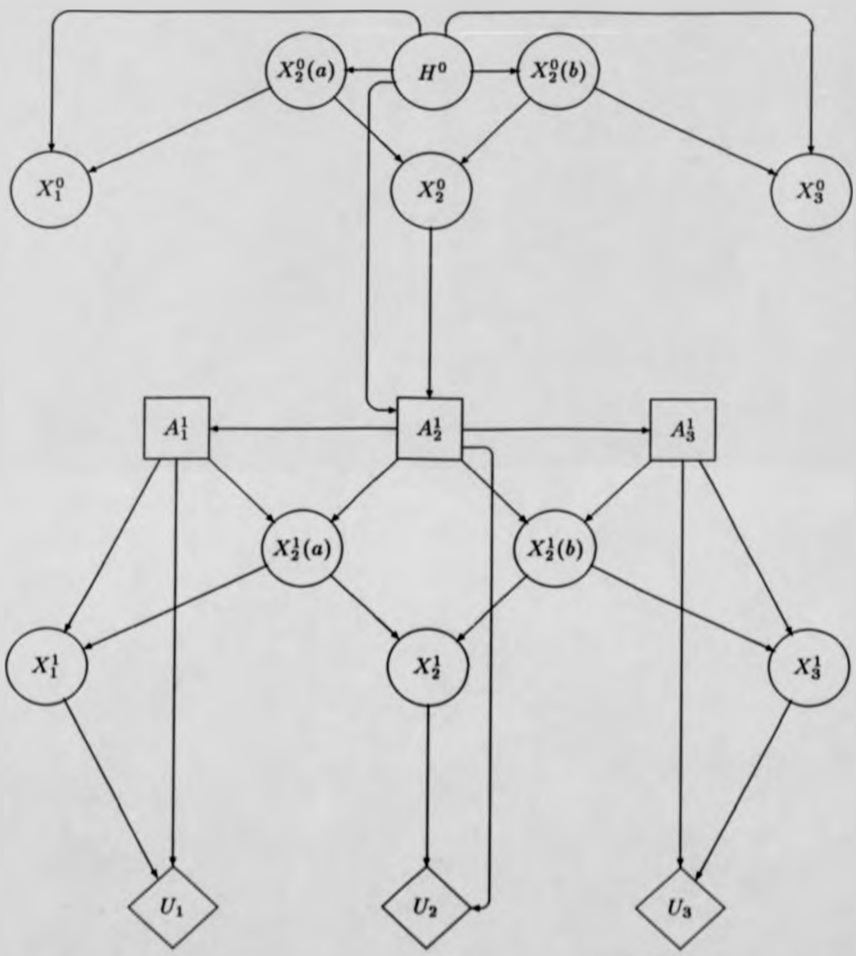


Figure 7.10 The simplified BID of a game played on a partially segmented market

Admittedly, this sort of analysis is a lot more difficult than anything we have considered in this thesis. This is due to the complexity which is inherent in the model, and which is represented in a BID such as that in figure 7.9. But just consider how much more difficult this type of analysis would be using other forms of modelling a game. A normal form of the 2-stage partially segmented market would be a table of figures in more than 20 dimensions. And even an extremely simplified extensive form might have one billion or more edges. In neither case would we be able to untangle even the simplest structural attributes. Using BIDs at least gives us a chance of doing so.

8 Conclusions and Suggestions for Further Research

8.1 Conclusions

We began by observing some of the problems associated with the modelling of games using the extensive and normal forms. At the end of chapter 2, I made the following statement:

> Given these fundamental problems with the modelling of games, and given that the relative primacy of the normal and extensive forms is at best undecided, we must surely conclude that there is room at least to consider alternative models. Any rival to these long-established models must demonstrate an advantage in some of those areas in which they are deficient. One area of deficiency common to both is the problem of modelling anything other than the simplest of situations. One possible way of overcoming this deficiency is by using an influence diagram instead of a tree as the basis of our model. This is the approach I propose to take.

By considering the final example in chapter 7, as shown in figure 7.9, we see that the BID is capable of modelling more complex situations than either the extensive or normal forms. Thus we have demonstrated that the BID may be worth considering as an alternative way of modelling games.

But there is much more to the BID than just a modelling system for complicated games. The foundations of the theory set out in chapter 4 provide for an entirely new area of application for the graphical model, and not simply an extension of some previous application. The BID allows us to model some of the most important qualitative aspects of a game, and then simplify them before even considering detailed quantitative features. Moreover, it provides an excellent framework for understanding the nature of a game, and thus the potential for a more intelligent and responsive analysis.

By making as few assumptions as were necessary to develop the theory, and by stating carefully at each stage and justifying those assumptions which were made, the results are consequently more powerful. The generality in which the BID was constructed and simplified means that this theory can apply to an extremely wide class of games. Indeed, it is possible to construct a model of almost any finite game, although I would not claim that any such model would necessarily be a useful one.

However, this thesis represents more than just a new way of modelling games; it involves a different approach to the theory of games itself. Given that the BID has been shown to have advantages over other modelling systems, we must conclude that the approach taken in this thesis, which led to the BID, is valid. It remains to be seen whether this approach and the BID modelling system will play a significant role in the future development of game theory.

8.2 Suggestions for Further Research

This thesis is primarily about foundations: the foundations of the BID; the foundations of rationality; the basic methods of model simplification. Hence there are many potentially fascinating areas of investigation which arise from the work done here, but are inevitably beyond the scope of this thesis. A few of these are listed here; doubtless there are many others.

1. Adapt the theory to encompass infinite games.

   Throughout this thesis, we have restricted our consideration to finite games. As was noted in chapter 4, the division between finite and infinite is perhaps the most substantial in qualitative terms; since the extension of our theory to infinite games is non-trivial, there has not been time to persue that line within the research for this thesis.

   Nevertheless, I can see no reason why in principle that extension could not be made. And the theory set out in chapter 7 presents some clues as to how this might be achieved, at least in the case of infinite repeated games. If it can be shown that a preference ordering exists with respect to experience in an infinite repeated game, then we need only consider locally optimal policies. Thus the infinite time horizon would not prevent an agent from determining some optimal policy in any given stage.

   However, there are some technical problems. For example, most of the proofs contained in this thesis rely in some way on the BID under consideration being finite. Thus to build a sound theory of infinite games, any adaptation of the current theory would very likely require some work on the foundational framework, and a fresh set of assumptions.

2. Weaken the common knowledge conditions which apply to the BID

   If there is no common knowledge BID, then given the current theory, all we can do is to use the joint BID (including every arc which appears in any individual agent's BID), and then only provided all agents agree on which set of nodes represents the set of objects in the game. It may, however, be possible to weaken this condition, and still obtain some useful results. For example, it might be interesting to consider what deductions could be made based on some subset of agents (possibly all those belonging to one or more actors) having some common knowledge BID.

Several problems would have to be solved in order to accomplish this. For instance, suppose two agents do not agree on a single BID representing a game. Even if we restrict our attention to the part of the BID on which they do agree, it is not clear that the simplifications of chapter 6 can be made with respect to that part without making some additional assumptions about the extent to which each agent acknowledges the disagreement. Furthermore, if this disagreement is in some way quantifiable, then Aumann's (1976) result may have implications as to whether the two agents could agree on the extent of their disagreement.

3. Combine the BID with the extensive form.

One attempt to combine the influence diagram with the decision tree is the contingent influence diagram of Fung and Shachter (1990). The analogues of these in game theory are the BID and extensive form respectively. It should be possible to combine these in a similar way. Such a framework would ideally maintain some of the advantages of both forms: the ability of the BID to model the conditional independence structure of a game, and the relative efficiency of the extensive form in representing asymmetric structures.

Our concept of sufficiency as it relates to the BID is perhaps most closely related to the concept of dominance which may be applied to simplify the extensive or normal forms. It follows that the repeated simplification of a BID based on sufficiency considerations is related to the iterated dominance or rationalisable principle advocated by Bernheim (1984) and Pearce (1984). Thus it may be possible to devise a system of alternating between the two methods to produce more substantial simplifications of a game than could be achieved by either method individually.

Needless to say, such a development would require a new foundational framework, which is well beyond the scope of this thesis.

4. Use the methodology to help analyse a practical problem.

   Queen (1994) has used DAGs to represent a competitive market, taking into consideration real data when designing the model. Such analysis can help a company to evaluate the market situation. By using the theory developed in this thesis, it ought to be possible to extend this analysis to include the decisions taken by the companies themselves.

   As was suggested at the end of chapter 7, there are many ways in which the use of BIDs might help a company to improve its decision making processes, and hence (hopefully) the quality of its decisions. The analysis of historic data in a market using the BID methodology may throw some light on whether a company's perceptions of the way in which other companies make decisions accurately reflect the true state of affairs.

   While such an analysis would undoubtedly be extremely hard to undertake, any results could have important implications for the strategic thinking which takes place within a company.

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